Error Estimations of Homotopy Perturbation Method for linear Integral and Integro-Differential Equations of the Third kind

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ABSTRACT

In this note, convex Homotopy perturbation method (HPM) is presented for the approximate solution of the linear Fredholm-Volterra integral and integro-differential equation. Convergence and rate of convergence of the HPM are proved for both equations. Five numerical examples are provided to verify the validity and accuracy of the proposed method. Example reveals that HPM is very accurate and simple to implement for integral and integro-differential equations.

INTRODUCTION

Integral equations occur naturally in many fields of science and engineering [1]. A computational approach to solve integral equation is an essential work in scientific research. Fredholm and Volterra integral equation is one of the most important integral equations.

Some iterative methods like Homotopy perturbation method (HPM) [2-11] and Adomian decomposition method (ADM) [12-14], Homotopy analysis method [15-22] have been applied to solve linear and nonlinear Fredholm and Volterra integral equations and integro-differential equations.

HPM has been used for a wide range of problems; for finding the exact and approximate solutions of nonlinear ordinary differential equations(ODEs) [7], linear and nonlinear integral equations [8,] the integro-differential equations [9,10] and the Volterra-Fredholm integral equations [11]. Hetmaniok et al. [23] have proposed method which is based on the homotopy perturbation method to solve Volterra–Fredholm integral equations of the second kind. The problem of the convergence of the series constructed is formulated and a proof of the formulation is given. Additionally, the estimation of the approximate solution is elaborated by taking the partial sums of the series. In 2016, Zulkarnain et al. [24] consider Fredholm-Volterro integro-differential equation (FVIDE) of order   of the third kind and solved it by using modified homotopy perturbation method (HPM). It is found that MHPM is a semi-analytical method and easy to apply for FVIDE. Numerical examples are given to present the efficiency and reliability of the method.In this note, we consider two types of integral equations:

$$s(x)u(x) = f(x) + \lambda_1 \int_a^b K_1(x,t)u(t)\mathrm{d}t + \lambda_2 \int_a^x K_2(x,t)u(t)\mathrm{d}t, \quad x \in [a,b], \tag{1}$$

$$s(x)u'(x) = f(x) + \lambda_1 \int_a^b K_1(x,t)u(t)\mathrm{d}t + \lambda_2 \int_a^x K_2(x,t)u(t)\mathrm{d}t, \quad x \in [a,b]. \tag{2}$$

Eqs. (1)-(2) are called Fredholm-Volterra integral equations (FVIEs) and Fredholm-Volterra integro-differential equations (FVIDEs) of the third kind respectively. For both type of equations the convergence of approximate solutions are established and four examples are provided to verify the accuracy of the proposed method.

The paper is organized as follows. In Section 2, we present the application of the HPM for the problem (1) and proof of the convergence. In Section 3, HPM is modified to solve the Eq. (2) and present the rate of convergence of the approximate solution. Section 4 provides four examples: first two examples related to the Eq. (1) and another two more examples dealt with Eq. (2). Numerical results show that the proposed methods are very accurate and converge fast. Conclusion is given in Section 5.

DERIVATION AND ESTIMATION OF THE APPROXIMATE METHOD FOR SOLVING FVIES OF THE THIRD KIND

To explain the homotopy perturbation method (HPM), we consider the general integral equation of the form

$$Lu + Nu = f \tag{3}$$

where L ia a linear operator and N is nonlinear operator.

As a possible remedy, we can define homotopy $H(v,p)$ by

$$H(v,p) = (1-p)F(v) + p(L(v) + N(v) - f) \tag{4}$$

where $F(v)$ is a functional operator with known solution u_0, which can be obtained easily. It is clear that, for

$$H(v,p) = 0 \tag{5}$$

from which we have

$$H(v,0) = F(v). \quad H(v,1) = Lv + Nv - f \tag{6}$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution $H(u,1)$. The embedding parameter p monotonically increases from zero to unit as the trivial problem $F(v) = 0$ is continuously deformed to the original problem $L(u) + N(u) - f = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [2].

To obtain the approximate solution of Eq.(3) we search solution in the series form

$$v(x) = \sum_{k=0}^{\infty} v_k(x)p^k. \tag{7}$$

When $p \to 1$, Eq.(5) corresponds to Eq. (3) and becomes the approximate solution i.e.,

$$u(x) = \lim_{p \to 1} v(x) = v_0 + v_1 + \dots \tag{8}$$

The series (8) is convergent for most cases, and the rate of convergence depends on $L(u)$ and $N(u)$.

Let us consider the linear Fredholm-Volterra integral equation (1) where the kernels $K_1, K_2 \in C([a,b] \times [a,b])$ and the functions $s, f \in C[a,b]$ are known, whereas the function u is to be determined. Define the operators $L(u)$ and $N(u)$ for Eq. (1) as follows

$$Lv(x) = s(x)v(x)$$

$$Nv(x) = -\lambda_1 \int_a^b K_1(x,t)v(t)\mathrm{d}t - \lambda_2 \int_a^x K_2(x,t)v(t)\,dt, \ v \in C[a,b]$$

with the assumption that $s(x) \neq 0$ for any $x \in [a,b]$. Rewriting Eq. (4) in the form

$$H(v,p) = Lv - u_0 + p(u_0 + Nv - f) \tag{9}$$

we solve the problem (1) as follows. For $p = 0$ the solution of the operator equation $H(v,0) = 0$ is equivalent to the solution of a trivial problem $s(x)v(x) - u_0(x) = 0$. For $p = 1$ the equation $H(v,1) = 0$ leads to the solution of Eq. (1).

By substituting the series solution (7) into the Eq. (5) and according to Abel''s Theorem the solution of Eq. (1) is obtained from the equation

$$L\left(\sum_{j=0}^{\infty} p^j v_j\right) = u_0 - p(u_0 - f) - N\left(\sum_{j=0}^{\infty} p^{j+1} v_j\right) \tag{10}$$

Comparing the same power of parameter p in both sides of Eq. (10), leads to the following relations

$$v_0 = \frac{u_0}{s} \tag{11}$$

$$v_1 = \frac{1}{s}(f - u_0 - Nv_0) \tag{12}$$

$$v_j = -\frac{1}{s}Nv_{j-1} \qquad j = 2,3,\ldots \tag{13}$$

According to the scheme (11)-(13) we define partial sum as follow

$$s_0(x) = \frac{u_0(x)}{s(x)}, \quad s_n(x) = \sum_{i=0}^{n} v_i(x) \tag{14}$$

Theorem 1 *thm1 Let* $K_1, K_2 \in C([a,b]\times[a,b])$ *and* $s, f \in C[a,b]$ *where* $s(x) \neq 0$ *for any* $x \in [a,b]$, *be continuous functions. In addition, if the following inequality*

$$\alpha = c(b-a)(|\lambda_1| \|K_1\| + |\lambda_2| \|K_2\|) < 1 \tag{15}$$

where $c = \left\| \frac{1}{s} \right\|$ *is satisfied and initial guess* u_0 *is chosen as a function continuous on the interval* $[a,b]$, *then series solution* (7) *is uniformly convergent to the exact solution* u *on the interval* $[a,b]$ *for each* $p = [0,1]$.

Proof. Prove of the Theorem 1 has not much changes of the Hetmaniok et al [23]. [Theorem 1].

Remark 1 *Hetmaniok et al.* [23] *[Remark 1]. In the thm1, the interval* $[a,b]$ *can be replaced by* $(a,b), (a,b]$ *or* $[a,b)$, *whereas the condition of continuity of* K_i *for* $i = \{1,2\}$ *and* f *in the appropriate regions* Ω_1 *and* Ω_2 *must be strengthened by adding the assumption of boundedness of these functions. Moreover, the conditions* $K_i \in C([a,b]\times[a,b])$ *can be replaced by some weaker condition, for example by the Lebesque integrability of kernel* K_i *on the set* $[a,b]\times[a,b]$ *and by the inequality* [25]

$$\int_a^b |K_i(x,t)|\,dt \leq M_i(b-a)$$

In the case of difficulties or impossibility of finding the sum of series (7) for $p = 1$, we may consider an approximate solution by taking the partial sum of the series (14). The first $n+1$ terms of series (14) in the limit $p \to 1$ create the so-called n th-order approximate solution in the form

$$s_n(x) = \sum_{k=0}^{n} v_k(x) \tag{16}$$

The solution $s_n(x)$ in (16) can be estimated on the basis of the following theorem. Theorem 2 is not much difference given in Hetmaniok et al. [23] [Theorem 2]

Theorem 2 *The error of* n *th-order approximate solution of (16) can be estimated by the inequality*

$$E_n \leq B\frac{\alpha^n}{1-\alpha}$$

where $E_n := \|v(x) - s_n(x)\|$, while $\|v_1\| \leq B$ and $|\alpha| < 1$.

Proof.

$$|v(x) - s_n(x)| = \left| \sum_{j=0}^{\infty} v_j(x) - \sum_{j=0}^{n} v_j(x) \right|$$

$$= \left| \sum_{j=n+1}^{\infty} v_j(x) \right| \leq \sum_{j=n+1}^{\infty} |v_j(x)| \leq \sum_{j=n+1}^{\infty} \|v_j\|$$

$$\leq \sum_{j=n+1}^{\infty} B\alpha^{j-1} = B\frac{\alpha^n}{1-\alpha}$$

DERIVATION AND ESTIMATION OF THE APPROXIMATE METHOD FOR SOLVING FVIDE OF THE THIRD KIND

Let us introduce the space of continuously differentiable functions $C^1([a,b])$ equipped with the norm

$$\|u\|_1 = \max_{a\leq x\leq b} |u(x)| + \max_{a\leq x\leq b} |u'(x)| \tag{17}$$

i.e.

$$\|u\|_1 = \|u\| + \|u'\|$$

where $\|\cdot\|$ is the standard norm in $C[a,b]$.

To apply convex HPM for FVIDEs (2) we rewrite it in the form

$$u'(x) = \frac{1}{s(x)}\left(f(x) + \lambda_1 \int_a^b K_1(x,t)u(t)\mathrm{d}t + \lambda_2 \int_a^x K_2(x,t)u(t)\mathrm{d}t \right), \quad u(a) = d_0 \tag{18}$$

where $u'(x)$ is the first order derivative of $u(x)$ with initial condition $u(a) = d_0$ and integrate both sides of Eq. (??) to yield

$$u(x) = d_0 + \int_a^x \frac{1}{s(t)} (f(t) + \lambda_1 \int_a^b K_1(t,\tau)u(\tau)\mathrm{d}\tau + \lambda_2 \int_a^t K_2(t,\tau)u(\tau)\mathrm{d}\tau)\mathrm{d}t \tag{19}$$

Write Eq. (19) in operator form

$$L_1 v = g_1 + N_1 v \tag{20}$$

where

$$L_1 v(x) = v(x), \quad g_1(x) = d_0 + \int_a^x \frac{f(t)}{s(t)} dt$$

$$N_1 v(x) = \int_a^x \frac{1}{s(t)} (\lambda_1 \int_a^b K_1(t,\tau)v(\tau)\mathrm{d}\tau + \lambda_2 \int_a^t K_2(t,\tau)v(\tau)\mathrm{d}\tau)\mathrm{d}t, \, v \in C^1[a,b] \tag{21}$$

Using the above definitions we obtain the homotopy operator for Eq.(??) as

$$H_1(v,p) = L_1 v - u_0 + p(u_0 - g_1 - N_1(v)) \tag{22}$$

where $p \in [0,1]$ is homotopy parameter and $u_0(x)$ defines as initial guess for Eq.(??). In a similar way, substituting the series solution (7) into equation

$$H_1(v,p) = 0$$

leads

$$L_1 \left(\sum_{j=0}^{\infty} p^j v_j \right) = u_0 + p(g_1 - u_0) - N_1 \left(\sum_{j=0}^{\infty} p^{j+1} v_j \right) \tag{23}$$

Comparing the expressions with the same power of parameter p, we obtain

$$v_0 = u_0 \tag{24}$$

$$v_1 = g_1 - u_0 - N_1 v_0 \tag{25}$$

$$v_j = -N_1 v_{j-1}, \qquad j = 2,3,\dots \tag{26}$$

Next, for the derivative of v we define

$$L_2 v(x) = v'(x), \quad g_2(x) = \frac{f(x)}{s(x)}$$

$$N_2 v(x) = -\frac{1}{s(x)} \left(\lambda_1 \int_a^b K_1(x,t)v(t)\mathrm{d}t + \lambda_2 \int_a^x K_2(x,t)v(t)\mathrm{d}t \right) \tag{27}$$

By using the homotopy perturbation method, the solution of operator equation

$$H_2(v,p) = L_2 v - u_0 + p(u_0 + N_2 v - g_2) = 0 \tag{28}$$

is searched in the form of power series

$$v'(x) = \sum_{j=0}^{\infty} p^j v_j'(x) \tag{29}$$

If the series (29) possesses a radius of convergence not greater than 1, then the series is absolutely convergent. By putting (29) into the Eq. (28) and comparing the expressions with the same power of parameter p leads to the relations

$$v_0' = u_0 \tag{30}$$

$$v_1' = g_2 - u_0 - N_2 v_0 \tag{31}$$

$$v_j' = -N_2 v_{j-1}, \qquad j = 2,3,\dots \tag{32}$$

The convergence of the approximate solution (7) of Eq. (2) is given in the Theorem 3..

Theorem 3 *Let $K_1, K_2 \in C([a,b] \times [a,b])$, s, $f \in C[a,b]$ be continuous function on the respective domain. In addition, if the inequality*

$$\beta = c(b-a)^2 (|\lambda_1| \|K_1\| + |\lambda_2| \|K_2\|) < 1 \tag{33}$$

where $c = \left\| \frac{1}{s} \right\|$, holds and the initial value u_0 is chosen as continuous function, then the series in Eqs. (7) is uniformly convergent to the exact solutions $u(x)$ in the sense of the C^1 norm on the interval $[a,b]$ for each $p \in [0,1]$.

Proof. Since the kernels K_1, K_2 and the function $f(x)$, $s(x)$ with $s(x) \neq 0$, $x \in [a,b]$, are continuous in the respective closed domain then

$$|K_1(x,t)| \leq \| K_1 \|, \quad |K_2(x,t)| \leq \| K_2 \|, \quad |f(x)| \leq \| f \|, \quad \left| \frac{1}{s(x)} \right| \leq \left\| \frac{1}{s} \right\| = c, \quad \forall x,t \in [a,b]$$

Let u_0 be chosen as continuous function and

$$\| u_0 \| = N_0$$

Then from the relations (24)-(26) it follows that

$$\| v_0 \| = \| u_0 \| = N_0$$

$$\| v_1 \| \leq | d_0 | + \| u_0 \| + c(b-a)(\| f \| + (b-a) \| v_0 \| (| \lambda_1 | \| K_1 \| + | \lambda_2 | \| K_2 \|))$$

$$\leq | d_0 | + N_0 + c(b-a) \| f \| + (b-a)N_0 \alpha := B$$

For the general case, from (26) we have the following estimation:

$$\| v_k \| \leq B(b-a)^{k-1} \alpha^{k-1}$$

$$\leq B \beta^{k-1} \tag{34}$$

where $\beta = (b-a)\alpha$.

From the series (7), for $p \in [0,1]$ it follows that

$$|v(x)| \leq \sum_{k=0}^{\infty} | p^k v_k(x) | \leq \sum_{k=0}^{\infty} \| v_k \| = \| v_0 \| + \sum_{k=1}^{\infty} \| v_k \|$$

$$\leq N_0 + B \sum_{k=1}^{\infty} \beta^{k-1}$$

$$= N_0 + \frac{B}{1-\beta} \tag{35}$$

Since the the estimate series (??) is the geometric series with the common ratio β and therefore it is convergent if common ratio $\beta < 1$. It implies that $v(x)$ is uniformly convergent on $[a,b]$ for each $p \in [0,1]$.

The assumptions for relations (30)-(32) implies

$$\| v_0' \| = \| u_0 \| = N_0$$

$$\| v_1' \| \leq \| u_0 \| + c(\| f \| + (b-a) \| v_0 \| (| \lambda_1 | \| K_1 \| + | \lambda_2 | \| K_2 \|))$$

$$\leq N_0 + c \| f \| + (b-a)N_0 \alpha = B'$$

$$\| v_k' \| \leq c((b-a) \| v_{k-1} \| (| \lambda_1 | \| K_1 \| + | \lambda_2 | \| K_2 \|)$$

$$\leq \| v_{k-1} \| \alpha.$$

Substituting (??) into the estimation of $v_k'(x)$ yields:

$$\| v_k' \| \leq B \beta^{k-2} \alpha,$$

$$= \frac{B}{(b-a)} \beta^{k-1}.$$

For the series in (29), we get

$$|v'(x)| \leq \sum_{k=0}^{\infty} | p^k v_k'(x) | \leq \sum_{k=0}^{\infty} \| v_k' \| \leq \| v_0' \| + \sum_{k=1}^{\infty} \| v_k' \|,$$

$$\leq \quad + \frac{}{(b \quad a)} \sum_{=1}^{\infty}$$

$$= N_0 + \frac{B}{(b-a)} \left(\frac{1}{1-\beta} \right). \tag{36}$$

The series $\sum_{k=1}^{\infty} \beta^k$ is the convergent geometric series possessing the common ratio $\beta < 1$. Hence, the solution $v(x)$ is differentiable. So that both series (7) and (29) are convergent if $| \beta | < 1$. Then according (17) we obtain convergence of the series (7) to the exact solutions $u(x)$ in the sense of the C^1 norm.

The partial sum $s_n(x)$ in Eq. (16) and its derivative can be estimated as follows.

Theorem 4 *The error E_n of nth-order approximate solution of (19) and its error of n th-order approximate derivative E_n' defined by (18) can be estimated by*

$$E_n \le B \frac{\beta^n}{1-\beta}, \quad E_n' \le \frac{B}{b-a}\frac{\beta^n}{1-\beta}$$

where $E_n := \| v(x) - s_n(x) \|$, $E_n' := \| v'(x) - s_{n'}(x) \|$, while $\| v_1 \| \le B$ and $|\beta| < 1$.

Proof.

$$| v(x) - s_n(x) | = \left| \sum_{j=n+1}^{\infty} v_j(x) \right| \le \sum_{j=n+1}^{\infty} \| v_j \|$$

$$\le \sum_{j=n+1}^{\infty} B\beta^{j-1} = B\frac{\beta^n}{1-\beta}.$$

and

$$| v'(x) - s_{n'}(x) | = \left| \sum_{j=n+1}^{\infty} v_j'(x) \right| \le \sum_{j=n+1}^{\infty} \| v_j' \|$$

$$\le \sum_{j=n+1}^{\infty} \frac{B}{b-a}\beta^{j-1} = \frac{B}{b-a}\frac{\beta^n}{1-\beta}.$$

EXAMPLES

Let the error estimate E_n and relative errors δ_n be defined by

$$E_n = \| u_d - s_n \| = \max_{a \le x \le b} | u_d - s_n |$$

and

$$= \frac{\| u_d \quad s_n \|}{\| \quad \|} \times 100\%$$

Example 1 *Consider the Fredholm-Volterra linear integral equation*

$$\frac{1}{x+1}u(x) = \frac{49}{45}x - \frac{1}{54}x^3 - 1 - \frac{1}{20}\int_0^2 xu(t)\,dt + \frac{1}{18}\int_0^x u(t)\,dt. \tag{37}$$

Solution. The exact solution is $u(x) = x^2 - 1$. We begin by verifying the conditions in Theorem 2. Since the kernels K_1, K_2 and functions $f(x), s(x)$ are continuous we need to check the inequality (15). For Eq. (37) we have

$$|\lambda_1| = \frac{1}{20}, |\lambda_2| = \frac{1}{18}, M_1 = \| K_1 \| = 2, M_2 = \| K_2 \| = 1, c = \left\| \frac{1}{s} \right\| = 3$$

Hence

$$\alpha = c(b-a)(|\lambda_1|M_1 + |\lambda_2|M_2) = \frac{14}{15} < 1$$

conditions of Theorem 1 holds, therefore HPM is convergent for any choice of initial guess. Let us choose $u_0(x) = \frac{1}{x+1}$, then calculating the successive functions v_j determined by relations (11)-(13) we obtain

$$v_0 = (x+1)u_0(x) = 1,$$

$$v_1 = (x+1)\left(-1 - \frac{1}{x+1} + \frac{47}{45}x - \frac{1}{54}x^3 \right),$$

$$v_2 = (x+1)\left(-\frac{61}{1350}x + \frac{1}{810}x^2 + \frac{47}{2430}x^3 - \frac{1}{3888}x^4 - \frac{1}{4860}x^5 \right),$$

Table 1 shows that the errors decreases very fast when number of terms n increases.

Table 1. Errors of the approximate solutions $s_n(x)$ defined by (16) for Eq. (37).

n	$E_n = \| u_d - s_n \|$	$\delta_n(\%)$	n	$E_n = \| u_d - s_n \|$	$\delta_n(\%)$
5	$4.17343915 \times 10^{-6}$	$1.39114638 \times 10^{-5}$	20	$7.30756701 \times 10^{-19}$	$2.43585567 \times 10^{-17}$
10	$2.35172667 \times 10^{-10}$	$7.83908886 \times 10^{-9}$	25	$4.07349003 \times 10^{-23}$	$1.35783001 \times 10^{-21}$
15	$1.31092835 \times 10^{-14}$	$4.36976116 \times 10^{-13}$	30	$2.27070391 \times 10^{-27}$	$7.56901305 \times 10^{-26}$

Example 2 *Consider the Fredholm-Volterra integral equation*

$$6e^{-x}u(x) = 6e^{-2x} - \frac{5}{18}xe^x + \frac{1}{9}\int_0^1 xe^{x+t}u(t)\,dt + \frac{1}{6}\int_0^x e^{x+t}u(t)\,dt \tag{38}$$

with the exact solution $u_d(x) = e^{-x}$.

Solution. Let initial guess $u_0 = 0$ be given. From Eq. (38) it follows that

$$\alpha = c(b-a)(|\lambda_1| M_1 + |\lambda_2| M_2) = \frac{e^2}{108} < 1$$

Parameter α in Eq. (38) satisfies the inequality (15).

From the **Table 2** it can be seen that the absolute and relative errors are decreased very fast when number of

Table 2. Errors of the approximate solutions $s_n(x)$ defined by (16) for Eq. (38)

n	$E_n = \|u_d - s_n\|$	$\delta_n(\%)$	n	$E_n = \|u_d - s_n\|$	$\delta_n(\%)$
5	$8.52615826 \times 10^{-8}$	$8.526158263 \times 10^{-6}$	20	$5.59658360 \times 10^{-30}$	$5.59658360 \times 10^{-28}$
10	$3.43859915 \times 10^{-15}$	$3.43859915 \times 10^{-13}$	25	$2.25784296 \times 10^{-37}$	$2.25784296 \times 10^{-35}$
15	$1.38724210 \times 10^{-22}$	$1.38724209 \times 10^{-20}$	30	$9.10886559 \times 10^{-45}$	$9.10886559 \times 10^{-43}$

terms n increases.

Example 3 *Consider the Fredholm-Volterra integro-differential equation of third kind*

$$(x+1)u'(x) = 1 + \frac{8}{9}x + \frac{1}{4}x^2 + \frac{1}{3}\int_0^1 xtu(t)\,dt + \frac{1}{2}\int_0^x u(t)\,dt \tag{39}$$

with the exact solution $u_d = x$ and initial condition $u(0) = 0$.

Solution. From Eq. (39) we have

$$|\lambda_1| = \frac{1}{3}, |\lambda_2| = \frac{1}{2}, M_1 = \|K_1\| = 1, M_2 = \|K_2\| = 1, c = \left\|\frac{1}{s}\right\| = 1, (b-a) = 1$$

Hence,

$$\beta = c(b-a)^2(|\lambda_1| M_1 + |\lambda_2| M_2) = \frac{5}{6} < 1$$

satisfies the conditions of Theorem 3. By choosing the initial function $u_0(x) = 1$ we obtain **(Table 3)**:

Table 3. Errors of the approximation solutions (16) for Eq. (39)

n	$E_n = \|u_d - s_n\|$	$\delta_n(\%)$	n	$E_n \|u_d - \hat{v}_n\|$	$\delta_n(\%)$
5	$1.96687746 \times 10^{-9}$	$1.96687746 \times 10^{-7}$	20	$7.97754087 \times 10^{-36}$	$7.97754087 \times 10^{-34}$
10	$4.75088504 \times 10^{-18}$	$4.75088504 \times 10^{-16}$	25	$3.82804414 \times 10^{-45}$	$3.82804414 \times 10^{-43}$
15	$1.02017261 \times 10^{-26}$	$1.02017261 \times 10^{-24}$	30	$1.81398763 \times 10^{-53}$	$1.81398763 \times 10^{-51}$

From these results we can conclude that the approximate solution converges very fast to the exact solution when n increases.

Example 4 *Consider the Fredholm-Volterra integro-differential equation of third kind*

$$\frac{4}{x+2}u'(x) = \frac{24x}{x+2} - \frac{2}{5} + \frac{1}{4}x - \frac{1}{20}x^6 - \frac{1}{18}x^4 + \frac{1}{5}\int_0^1 (1-xt)u(t)\,dt - \frac{1}{3}\int_0^x (xt^2 - x^2t)u(t)\,dt, \quad u(0) = 1 \tag{40}$$

Solution. Exact solution of Eq. (40) is $u_d = 1 + 3x^2$. It is not difficult to see that

$$\beta = c(b-a)^2(|\lambda_1| M_1 + |\lambda_2| M_2) = \frac{11}{40} < 1$$

satisfies the condition of Theorem 3.

The results in **Table 4**, show that the absolute and relative errors of Eq. (40) decreases very fast with initial guess $u_0(x) = x + 2$.

Table 4. Errors of the approximation solutions (16) for Eq. (40)

n	$E_n = \|u_d - \hat{v}_n\|$	$\delta_n(\%)$	n	$E_n = \|u_d - s_n\|$	$\delta_n(\%)$
5	$5.79771528 \times 10^{-7}$	$1.44942882 \times 10^{-5}$	20	$4.38679146 \times 10^{-27}$	$1.09669786 \times 10^{-25}$
10	$1.13820421 \times 10^{-13}$	$2.84551053 \times 10^{-12}$	25	$8.61212628 \times 10^{-34}$	$2.15303157 \times 10^{-32}$
15	$2.23451662 \times 10^{-20}$	$5.58629154 \times 10^{-19}$	30	$1.69072818 \times 10^{-40}$	$4.22682043 \times 10^{-39}$

Example 5 *Consider the Fredholm-Volterra integral equation of the third kind in the form*

$$\frac{5}{x+2}u(x) = \frac{5x^2}{x+2} - \frac{1}{3}x^2 - \frac{1}{24}x^4 + \frac{1}{8}\int_0^2 x^2 u(t)\,dt + \frac{1}{2}\int_0^x (x-t)u(t)\,dt \tag{41}$$

Solution. The exact solution is $u(x) = x^2$ with the initial guess $u_0(x) = \dfrac{2}{x+2}$. In this example, the conditions of the Theorem 2 fails. Since

$$| \lambda_1 | = \frac{1}{8}, | \lambda_2 | = \frac{1}{2}, M_1 = \| K_1 \| = \frac{1}{2}, M_2 = \| K_2 \| = 1, c = \| \frac{1}{s} \| = \frac{4}{5}$$

hence

$$\alpha = c(b-a)(| \lambda_1 | M_1 + | \lambda_2 | M_2) = \frac{6}{5} > 1$$

From the **Table 5**, we can conclude that approximate solution for Eq. (41) would converges to the exact solution but it might depends on the choice of the initial guess.

Table 5. Errors of the approximation solutions (16) for Eq. (41)

n	$E_n = \| u_d - \hat{v}_n \|$	(%)	n	$E_n = \| u_d - s_n \|$	$\delta_n (\%)$
5	$6.01228713 \times 10^{-3}$	0.15030718	20	$5.09091226 \times 10^{-11}$	$1.27272807 \times 10^{-9}$
10	$1.22543958 \times 10^{-5}$	$3.06359894 \times 10^{-4}$	25	$1.03764182 \times 10^{-13}$	$2.59410456 \times 10^{-12}$
15	$2.49772003 \times 10^{-8}$	$6.24430008 \times 10^{-7}$	30	$2.11494619 \times 10^{-16}$	$5.28736548 \times 10^{-15}$

CONCLUSION

In this note, we have analyzed HPM for solving linear FVIEs and FVIDEs of the third kind. From the Theorem 2 it follows that HPM for FVIEs converges uniformly if $\alpha < 1$. Meanwhile, HPM for FVIDE is uniformly convergent and differentiable if $\beta < 1$.

Examples 1-2 corresponding **Tables 1 and 2** verify that HPM is very accurate and stable for the FVIEs. Meanwhile Examples 3-4 show that HPM is convergent to the exact solution very fast when number of points n increases. Example 5, shows that convex HPM converges to the exact solution by the suitable choice of initial guess even though conditions of Theorem 2 is not satisfied.

From the examples above, we can conclude that HPM method for solving FVIE and FVIDE converge very fast when number of point n increases and given functions and kernels are satisfied the certain conditions in thm1 and thm3.

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