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On $\mathbb{Z}_N$-Invariant Subgroups of Semi-Simple Lie Groups

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ABSTRACT

We employ Mathematica to find $\mathbb{Z}_N$-invariant subgroups of E_8 for application in M-theory. These $\mathbb{Z}_N$-invariant subgroups are phenomenologically important and in some cases they resemble the gauge groups of our real world. We present a specific example of $\mathbb{Z}_N$-invariant subgroups of E_8, which turn up in orbifold compactification of M-theory. However, the procedure can be applied for any $\mathbb{Z}_N$ group that acts by shifts (translations) in the root lattice of semisimple Lie groups with A_n, B_n, C_n, D_n, E_6, E_7 and E_8 factors. PACS: 02.20. Rt, 11.25.Mj.

INTRODUCTION

In models where part of spacetime is compactified, the geometry of compact space affects the gauge symmetries of the model. Herein, we consider the Horava-Witten M-theory [1,2], where the 11th dimension is compactified to an interval, I, and there are two ten-dimensional planes at the boundaries of I. It is convenient to identify $I = S^1/Z_2$, acting as $\mathbb{Z}_2 : \phi \to -\phi$, so that the boundary of I consists of the fixed points of this $\mathbb{Z}_2$-action. On each one of these ten-dimensional spacetime planes there is an independent copy of E_8 gauge fields (principal vector bundle). To produce considerably more realistic models with 4-dimensional spacetime, one may proceed as follows:

1. Impose twisted periodicity conditions on six of the ten dimensions of the boundary spacetime planes, passing $\mathbb{R}^6 \to (T^6/\Delta) = ((\mathbb{R}^6/\Lambda)/\Delta)$, where Λ is a suitable 6-dimensional lattice and Δ is a symmetry of Λ. We consider $\Delta = \mathbb{Z}_N$.

2. Simultaneously embed the Δ action into the E_8 structure group of the gauge fields on each of the two boundary spacetimes, the structure groups are broken to subgroups of E_8 that are invariant with respect to the Δ-action.

This is referred to as "compactifying the Horava-Witten M-theory on a T^6/Δ orbifold", and Δ is the "orbifold group." Typically, Δ acts by rotations on the compact space coordinates, and at the same time by shifts (translations) in the E_8 root lattice [3, 4].

In Ref [5], we have constructed $\mathbb{Z}_7$-orbifold models in M-theory. We used Mathematica to find the $\mathbb{Z}_7$-invariant subgroups of E_8. In this paper we present the details of the Mathematica computation codes and the procedure that we have used in Ref [5]. This procedure may be used, perhaps with minor adaptations, for higher order (iterated) orbifolds as well, and in situations where one needs to find the $\mathbb{Z}_7$-invariant subgroups of any of the semisimple Lie groups with A_n, B_n, C_n, D_n, E_6, E_7 and E_8 factors, where $\mathbb{Z}_N$ acts by shifts in the root lattice.

THE ALGORITHM

Consider the root lattice $\mathcal{P}$ of one of the simple Lie algebras $\mathfrak{g}=A_n, B_n, C_n, D_n, E_6, E_7$ or E_8 Let u denote a shift (translation) vector in $\mathcal{P}$ acting as $e^{2\pi i v\cdot \mathbf{p}}|\mathbf{p}\rangle$, on $|\mathbf{p}\rangle \in P$ [3, 4]; require moreover that $(e^{2\pi i v\cdot \mathbf{p}})^N = 1$, so that v generates a $\mathbb{Z}_N$ action on $\mathcal{P}$, and thus on G. The root vectors of g that are invariant with respect to this u-action

$$e^{2\pi i v\cdot \mathbf{p}}|\mathbf{p}\rangle = |\mathbf{p}\rangle, \quad \mathbf{p} \in \mathcal{P} \tag{1.1}$$

are the root vectors of a subgroup $H_I \subset G$ that is invariant with respect to the $\mathbb{Z}_N$ -action generated by v. Different shift vectors v define different $\mathbb{Z}_N$-actions, and therefore different $\mathbb{Z}_N$-invariant subgroups of G. Upon identifying those that are equivalent by G-conjugation, we find the inequivalent $\mathbb{Z}_N$-invariant subgroups, H_I, for I = 1,2,... Without loss of generality, we restrict the $[\frac{k}{N}(\mathrm{mod}1)]$-valued components of v in (1.1) to the standard range $\{0, \frac{1}{N}, \frac{2}{N}, \cdots, \frac{N-1}{N}\}$.

Note: As the so-defined $\mathbb{Z}_N$-invariant subgroups $H_I \subset G$ are explicitly defined in terms of the root lattice of G, they are by definition regular [6-13]. In addition, the condition (1.1) is trivially satisfied for the zero-weight vectors $\mathbf{p}_C = \{0,0,\ldots,0\}$ corresponding to Cartan generators of G. Therefore,

$$rank\,(H_I) = rank\,(G), \tag{2}$$

and all so-defined $\mathbb{Z}_N$ -invariant regular subgroups of G also have maximal rank.

Step 1: Find the set of positive root vectors of G, denoted w.

Step 2: Based on the above restrictions, we construct all possible $\mathbb{Z}_N$ shift vectors v.

Step 3: Find all the subgroups[1] $H_I \subset G$.

Step 4: For each one of the subgroups $H_I \subset G$, define the following four variables:

t is the set of positive root vectors of $H_I \subset G$;

p : = |t| is total number of positive root vectors in $H_I \subset G$;

r : = $\mathrm{rank}_I(H_I)$, defined as the rank of semisimple part of $H_I \subset G$, i.e., without U(1)-factors; m is the number of A_1 factors, if any, in $H_I \subset G$.

These three variables can be read off by looking at the subgroup H_I and can be used as identifiers of the group. If these three variables do not suffice to identify $H_I \subset G$ unambiguously, define another variable:

m_2 is the number of A_2 factors in H_I, if any.

If {p, r, m, m_2} turns out not to suffice to identify $H_I \subset G$ unambiguously, we look for A_3, A_4. . . factors in H_I, the numbers of which, m_3, m_4...,will be necessary to identify $H_I \subset G$ unambiguously.

Step 5: Pick the first u from Step 2.

Step 5a : Set t = ∅. For all $\mathbf{w}_a \in \mathbf{W}$, if[2] $\mathbf{v}\cdot\mathbf{w}_a = \mathbb{Z}$, append $\mathbf{w}_a$ into the set t.

Step 5b: Compute {p, r, m, ...} of this t (see Section 5 for the procedure).

Step 5c: Identify the subgroup $H_I \subset G$ by comparing {p, r, m, ...} with the list from Step 4.

Step 6: Pick the next from Step 2, and go to Step 5a.

Steps 1–4 are preparatory. In particular, Step 4 sets up the string of identifiers {p, r, m, m_2, ...} as an "address" of the regular subgroups HI (I = 1, 2, 3, . . .) of a given simple Lie group G. For the purposes of specific applications, such as in M-theory [2,5,14] with G = E_8 and $\mathbb{Z}_N$ acting by translations in the root lattice (1), a subset of the identifiers {p, r, m, m_2,...} sufficed.

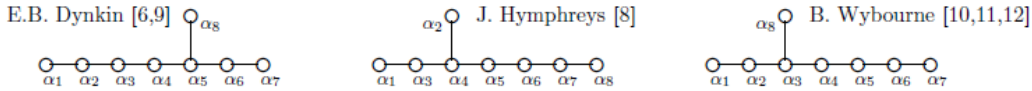
ROOTS AND SHIFT VECTORS

We take the adjoint representation of the group G and calculate its positive root vectors from the highest root using the standard algorithm [6,8-,12]. Take for example the group G = E_8. Any concrete representation of these roots will depend on a choice of a basis, and there exist at least three fairly standard conventions, corresponding to the labeling of nodes of the Dynkin diagram of E_8, as shown[3] in **Figure 1.** Being interested primarily in high energy physics applications such as in Ref [2,5,14], we follow the conventions of Refs [10,11], which provide the decades-long standard in the high energy physics.

[1]For all simple Lie groups of rank ≤ 8 and several of higher rank, the maximal subgroups are listed in the literature [6,11,13];

[2]Since the root lattice shift v corresponds to a generator $g(\mathbf{v}) \in \mathbb{Z}_N$ so that all elements of $\mathbb{Z}_N$ are powers of $g(v)$, it follows that root vectors satisfying $\mathbf{v}\cdot\mathbf{w}_a = \mathbb{Z}$ are in fact invariant with respect to all of $\mathbb{Z}_N$.

[3]To save space, negative root vector components are denoted by an over-bar: $\bar{1} \overset{def}{=} -1$, $\bar{2} \overset{def}{=} -2$, , *etc.*



Adjoint: {1,0,0,0,0,0,0,0}   Adjoint: {0,0,0,0,0,0,0,1}   Adjoint: {0,0,0,0,0,0,1,0}

$$\begin{bmatrix}2&\bar{1}&0&0&0&0&0&0\\ \bar{1}&2&\bar{1}&0&0&0&0&0\\ 0&\bar{1}&2&\bar{1}&0&0&0&0\\ 0&0&\bar{1}&2&\bar{1}&0&0&0\\ 0&0&0&\bar{1}&2&\bar{1}&0&\bar{1}\\ 0&0&0&0&\bar{1}&2&\bar{1}&0\\ 0&0&0&0&0&\bar{1}&2&0\\ 0&0&0&0&\bar{1}&0&0&2\end{bmatrix} \quad \begin{bmatrix}2&0&\bar{1}&0&0&0&0&0\\ 0&2&0&\bar{1}&0&0&0&0\\ \bar{1}&0&2&\bar{1}&0&0&0&0\\ 0&\bar{1}&\bar{1}&2&\bar{1}&0&0&0\\ 0&0&0&\bar{1}&2&\bar{1}&0&0\\ 0&0&0&0&\bar{1}&2&\bar{1}&0\\ 0&0&0&0&0&\bar{1}&2&\bar{1}\\ 0&0&0&0&0&0&\bar{1}&2\end{bmatrix} \quad \begin{bmatrix}2&\bar{1}&0&0&0&0&0&0\\ \bar{1}&2&\bar{1}&0&0&0&0&0\\ 0&\bar{1}&2&\bar{1}&0&0&0&\bar{1}\\ 0&0&\bar{1}&2&\bar{1}&0&0&0\\ 0&0&0&\bar{1}&2&\bar{1}&0&0\\ 0&0&0&0&\bar{1}&2&\bar{1}&0\\ 0&0&0&0&0&\bar{1}&2&0\\ 0&0&\bar{1}&0&0&0&0&2\end{bmatrix}$$

Figure 1. The Dynkin diagram, the highest root of the adjoint representation and the Cartan matrix of E_8, given in three fairly standard conventions and some corresponding references.

The highest root of the irreducible (248-dimensional) adjoint representation of E_8 is {0, 0, 0, 0, 0, 0, 1, 0 }. The entire root system can be obtained from the highest root by subtracting from it the positive simple root vectors as follows: in any given root vector w, a positive value of the nth component, w[n], indicates the number of times the nth positive simple root α_n can be subtracted from w minus the number of times α_n can be added to w so as to get another root or zero [10,12]. For example, $\alpha_1 = \{2, \bar{1}, 0, 0, 0, 0, 0, 0\}$ is the first positive simple root (and the 1st row in the Cartan matrix; **Figure 1**); it may be subtracted from itself twice[4], producing:

$$\{2,\bar{1},0,0,0,0,0,0\} \xrightarrow{-\alpha_1} \{0,0,0,0,0,0,0,0\} \xrightarrow{-\alpha_1} \{2,\bar{1},0,0,0,0,0,0\}. \tag{3.1}$$

All three of these vectors are indeed in the root system of E_8. Starting with $\lambda = \{0,0,0,0,0,0,1,0\}$, the positive simple root $\alpha_7 = \{0,0,0,0,0,\bar{1},2,0\}$ may be subtracted once (since λ is the highest root, no positive root can be added and still get a root):

$$\{0,0,0,0,0,0,1,0\} \xrightarrow{-\alpha_7} \{0,0,0,0,0,1,1,0\}, \tag{3.2}$$

Where upon $\alpha_6 = \{0,0,0,0,\bar{1},2,\bar{1},0\}$ may be subtracted once

$$0,0,0,0,0,1,\bar{1},0\} \rightarrow \{0,0,0,0,1,\bar{1},0,0\}, \tag{3.3}$$

Proceeding in this way halts with $\{0,0,0,0,0,0,\bar{1},0\}$, having produced 240 (nonzero) root vectors and eight copies of $\{0,0,0,0,0,0,0,0\}$. Jointly, they span the 248-dimensional adjoint representation of E_8.

In fact, every finite-dimensional unitary representation of any semisimple Lie group may be represented in a similar way: We recall that all such representations are spanned by weight vectors that are determined by a highest weight from which all others are obtained by iteratively subtracting the positive simple roots as outlined above; see Refs.[1-12,8]. By definition of λ being the highest weight, no positive simple root may be added to it and get a vector within the weight system of λ. Therefore, a positive nth component λ [n] >0 in the highest weight λ necessarily means that α_n may be subtracted λ [n] >0 number of times from λ; plot the so-obtained "α_n-descendants of λ," $(\lambda - k\alpha_n)$, k levels below λ. Now proceed downward level by level, seeking an $m^{th} \neq n^{th}$ positive component in an α_n-descendant weight of λ, from which to construct α_m-decendants. Starting from a level where the $m^{th} \neq n^{th}$ component $(\lambda - k\alpha_n)[m] > 0$ but $(\lambda - k\alpha_n + \alpha_m)$ is not in the weight-system (immediately above $(\lambda - k\alpha_n)$) implies that α_m can be subtracted from $(\lambda - k\alpha_n)$ precisely $(\lambda - k\alpha_n)[m]$ number of times, producing a chain of α_m-descendants.

Proceeding in this fashion eventually terminates and generates the complete weight system when starting from highest weights λ that define finite-dimensional representations [7,8,10-12]. Of course, one can just as easily start from the lowest weight and add the positive simple roots in the analogous fashion. In the special case of the adjoint representation, which is our focus at present, the nonzero weight vectors are called root vectors instead.

By plotting the weights (roots) below those from which they are obtained by subtracting positive simple roots and connecting them by arrows (for illustration, see (3.6) below), we obtain a "spindle shaped" graph called the weight (root) diagram of the (adjoint) representation. In the root diagram of the adjoint representation of a group G of rank r, the middle row of the root diagram is populated by r copies of {0,...,0}, representing the r Cartan generators. The row immediately above the middle is populated by the r positive simple root vectors; the roots above the middle row are the positive root vectors of G, while the roots below the middle row are the negative root vectors and are the sign-reversed copies of the positive root vectors. For every Lie group and its algebra, it therefore suffices to map out the subsystem of positive roots.

E_8: The following Mathematica code computes the 120 positive root vectors of E_8 following the conventions of Ref [10,11]. The code is adapted to any other convention by changing the basis for both the Cartan matrix and the highest root, i.e., the

[4]To be meticulous, the fact that the first component of $\alpha_1 = \{2,\bar{1},0,0,0,0,0,0\}$ is $\boldsymbol{\alpha}_1[1] = +2$ merely means that we can subtract $\boldsymbol{\alpha}_1$ from itself two more times than we can add $\boldsymbol{\alpha}_1$ to itself, and still get a root or zero. However, since $\alpha_1 + \alpha_1 \neq 0$ can be shown not to be a root, it follows that $\boldsymbol{\alpha}_1$ can be added to itself zero number of times while staying in the root system, and so can be subtracted from itself precisely two times.

Mathematica variables a and g[0], respectively; for those displayed in **Figure 1**, a simple permutation of columns and rows will suffice. Also, we use "external/global" variable arrays so that the intermediate computations are all accessible, e.g., for troubleshooting and for tracing the functioning of the code; it is then necessary to start with clearing the required symbols, listed explicitly for each code. The Reader may also find the global command ClearAll ["Global'*"] useful, which clears all user-defined variables from previous computations.

Input (1)

ClearAll [a, g, e];

$$
\begin{aligned}
a \;=\; \{&\{2, \bar{1}, 0, 0, 0, 0, 0, 0\},\\
&\{\bar{1}, 2, \bar{1}, 0, 0, 0, 0, 0\},\\
&\{0, \bar{1}, 2, \bar{1}, 0, 0, 0, \bar{1}\},\\
&\{0, 0, \bar{1}, 2, \bar{1}, 0, 0, 0\},\\
&\{0, 0, 0, \bar{1}, 2, \bar{1}, 0, 0\},\\
&\{0, 0, 0, 0, 0, \bar{1}, 2, 0\},\\
&\{0, 0, \bar{1}, 0, 0, 0, 0, 2\}\};
\end{aligned}
$$

```
g [0] = {{0, 0, 0, 0, 0, 0, 1, 0}};

g [1] = Table[ Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[p]]]],

{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}];

e[x_] : =  e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]]  ==  1, g[x][[j]]- a[[i]],

If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]]], {i, 8}], {j, Length[g[x]]}],

Table[Table[ If[g[x- 1][[l]][[k]]  == 2, g[x- 1][[l]]- 2 a[[k]]], {k, 8}],

{l, Length[g[x- 1]]}]}, 2]];

g[x_] : = g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];

Flatten[Table[g[m], {m, 0, 28}], 1])
```

Output (1)

$$
\begin{gathered}
\{\{0,0,0,0,0,0,1,0\},\{0,0,0,0,0,1,\bar{1},0\},\{0,0,0,0,1,\bar{1},0,0\},\{0,0,0,1,\bar{1},0,0,0\},\{0,0,1,\bar{1},0,0,0,0\},\\
\{0,1,\bar{1},0,0,0,0,1\},\{0,1,0,0,0,0,0,\bar{1}\},\{1,\bar{1},0,0,0,0,0,1\},\{\bar{1},0,0,0,0,0,0,1\},\{1,\bar{1},1,0,0,0,0,\bar{1}\},\\
\{\bar{1},0,1,0,0,0,0,\bar{1}\},\{1,0,\bar{1},1,0,0,0,0\},\{\bar{1},1,\bar{1},1,0,0,0,0\},\{1,0,0,\bar{1},1,0,0,0\},\{\bar{1},1,0,\bar{1},1,0,0,0\},\\
\{0,\bar{1},0,1,0,0,0,0\},\{1,0,0,0,\bar{1},1,0,0\},\{\bar{1},1,0,0,\bar{1},1,0,0\},\{0,\bar{1},1,\bar{1},1,0,0,0\},\{1,0,0,0,0,\bar{1},1,0\},\\
\{\bar{1},1,0,0,0,\bar{1},1,0\},\{0,\bar{1},1,0,\bar{1},1,0,0\},\{0,0,\bar{1},0,1,0,0,1\},\{1,0,0,0,0,0,\bar{1},0\},\{\bar{1},1,0,0,0,0,\bar{1},0\},\\
\{0,\bar{1},1,0,0,\bar{1},1,0\},\{0,0,\bar{1},1,\bar{1},1,0,1\},\{0,0,0,0,1,0,0,\bar{1}\},\{0,\bar{1},1,0,0,0,\bar{1},0\},\{0,0,\bar{1},1,0,\bar{1},1,1\},\\
\{0,0,0,\bar{1},0,1,0,1\},\{0,0,0,1,\bar{1},1,0,\bar{1}\},\{0,0,\bar{1},1,0,0,\bar{1},1\},\{0,0,0,\bar{1},1,\bar{1},1,1\},\{0,0,0,1,0,\bar{1},1,\bar{1}\},\\
\{0,0,1,\bar{1},0,1,0,\bar{1}\},\{0,0,0,\bar{1},1,0,\bar{1},1\},\{0,0,0,0,\bar{1},0,1,1\},\{0,0,0,1,0,0,\bar{1},\bar{1}\},\{0,0,1,\bar{1},1,\bar{1},1,\bar{1}\},\\
\{0,1,\bar{1},0,0,1,0,0\},\{0,0,0,0,\bar{1},1,\bar{1},1\},\{0,0,1,\bar{1},1,0,\bar{1},\bar{1}\},\{0,0,1,0,\bar{1},0,1,\bar{1}\},\{0,1,\bar{1},0,1,\bar{1},1,0\},\\
\{1,\bar{1},0,0,0,1,0,0\},\{\bar{1},0,0,0,0,1,0,0\},\{0,0,0,0,0,\bar{1},0,1\},\{0,0,1,0,\bar{1},1,\bar{1},\bar{1}\},\{0,1,\bar{1},0,1,0,\bar{1},0\},\\
\{0,1,\bar{1},1,\bar{1},0,1,0\},\{1,\bar{1},0,0,1,\bar{1},1,0\},\{\bar{1},0,0,0,1,\bar{1},1,0\},\{0,0,1,0,0,\bar{1},0,\bar{1}\},\{0,1,\bar{1},1,\bar{1},1,\bar{1},0\},\\
\{0,1,0,\bar{1},0,0,1,0\},\{1,\bar{1},0,0,1,0,\bar{1},0\},\{1,\bar{1},0,1,\bar{1},0,1,0\},\{\bar{1},0,0,0,1,0,\bar{1},0\},\{\bar{1},0,0,1,\bar{1},0,1,0\},\\
\{0,1,\bar{1},1,0,\bar{1},0,0\},\{0,1,0,\bar{1},0,1,\bar{1},0\},\{1,\bar{1},0,1,\bar{1},1,\bar{1},0\},\{1,\bar{1},1,\bar{1},0,0,1,0\},\{\bar{1},0,0,1,\bar{1},1,\bar{1},0\},\\
\{\bar{1},0,1,\bar{1},0,0,1,0\},\{0,1,0,\bar{1},1,\bar{1},0,0\},\{1,\bar{1},0,1,0,\bar{1},0,0\},\{1,\bar{1},1,\bar{1},0,1,\bar{1},0\},\{1,0,\bar{1},0,0,0,1,1\},\\
\{\bar{1},0,0,1,0,\bar{1},0,0\},\{\bar{1},0,1,\bar{1},0,1,\bar{1},0\},\{\bar{1},1,\bar{1},0,0,0,1,1\},\{0,1,0,0,\bar{1},0,0,0\},\{1,\bar{1},1,\bar{1},1,\bar{1},0,0\},\\
\{1,0,\bar{1},0,0,1,\bar{1},1\},\{1,0,0,0,0,0,1,\bar{1}\},\{\bar{1},0,1,\bar{1},1,\bar{1},0,0\},\{\bar{1},1,\bar{1},0,0,1,\bar{1},1\},\{\bar{1},1,0,0,0,0,1,\bar{1}\},\\
\{0,\bar{1},0,0,0,0,1,1\},\{1,\bar{1},1,0,\bar{1},0,0,0\},\{1,0,\bar{1},0,1,\bar{1},0,1\},\{1,0,0,0,0,1,\bar{1},\bar{1}\},\{\bar{1},0,1,0,\bar{1},0,0,0\},\\
\{\bar{1},1,\bar{1},0,1,\bar{1},0,1\},\{\bar{1},1,0,0,0,1,\bar{1},\bar{1}\},\{0,\bar{1},0,0,0,1,\bar{1},1\},\{0,\bar{1},1,0,0,0,1,\bar{1}\},\{1,0,\bar{1},1,\bar{1},0,0,1\},\\
\{1,0,0,0,1,\bar{1},0,\bar{1}\},\{\bar{1},1,\bar{1},1,\bar{1},0,0,1\},\{\bar{1},1,0,0,1,\bar{1},0,\bar{1}\},\{0,\bar{1},0,0,1,\bar{1},0,1\},\{0,\bar{1},1,0,0,1,\bar{1},\bar{1}\},\\
\{0,0,\bar{1},1,0,0,1,0\},\{1,0,0,\bar{1},0,0,0,1\},\{1,0,0,1,\bar{1},0,0,\bar{1}\},\{\bar{1},1,0,\bar{1},0,0,0,1\},\{\bar{1},1,0,1,\bar{1},0,0,\bar{1}\},\\
\{0,\bar{1},0,1,\bar{1},0,0,1\},\{0,\bar{1},1,0,1,\bar{1},0,\bar{1}\},\{0,0,\bar{1},1,0,1,\bar{1},0\},\{0,0,0,\bar{1},1,0,1,0\},\{1,0,1,\bar{1},0,0,0,\bar{1}\},\\
\{\bar{1},1,1,\bar{1},0,0,0,\bar{1}\},\{0,\bar{1},1,\bar{1},0,0,0,1\},\{0,\bar{1},1,1,\bar{1},0,0,\bar{1}\},\{0,0,\bar{1},1,1,\bar{1},0,0\},\{0,0,0,\bar{1},1,1,\bar{1},0\},\\
\{0,0,0,0,\bar{1},1,1,0\},\{1,1,\bar{1},0,0,0,0,0\},\{\bar{1},2,\bar{1},0,0,0,0,0\},\{0,\bar{1},2,\bar{1},0,0,0,\bar{1}\},\{0,0,\bar{1},0,0,0,0,2\},\\
\{0,0,\bar{1},2,\bar{1},0,0,0\},\{0,0,0,\bar{1},2,\bar{1},0,0\},\{0,0,0,0,\bar{1},2,\bar{1},0\},\{0,0,0,0,0,\bar{1},2,0\},\{2,\bar{1},0,0,0,0,0,0\}\}
\end{gathered}
$$

Replacing Flatten [Table[g[m], {m,0,28}], 1] ⟶ Do[Print[g[m]], {m,0,58}] in the last line of input (1) prints all the roots, at their actual level and produces the characteristic spindle-shaped listing.

E_7: For E_7 and E_6 the input codes are similar. For E_7, the highest root of the adjoint representation, 133, is {1,0,0,0,0,0,0}. Its (133−7)/2 = 63 positive root vectors of E_7 are found by the following code:

Input (2)

```
ClearAll [a, e, g];
```

$$a = \{\{2, \bar{1}, 0, 0, 0, 0, 0, 0\}, \\ \{\bar{1}, 2, \bar{1}, 0, 0, 0, 0, 0\}, \\ \{0, \bar{1}, 2, \bar{1}, 0, 0, 0, \bar{1}\}, \\ \{0, 0, \bar{1}, 2, \bar{1}, 0, 0, 0\}, \\ \{0, 0, 0, \bar{1}, 2, \bar{1}, 0, 0\}, \\ \{0, 0, 0, 0, 0, \bar{1}, 2, 0\}, \\ \{0, 0, \bar{1}, 0, 0, 0, 0, 2\}\};$$

```
g[0] = {{1, 0, 0, 0, 0, 0, 0}};
g[1] = Table[ Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[p]]]],
{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}];
e[x_] : =  e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]] == 1, g[x][[j]]- a[[i]],
If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]], {i, 7}], {j, Length[g[x]]}],
Table[Table[ If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, 7}],
{l, Length[g[x- 1]]}]}, 2]];
g[x_] : =  g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];
Flatten[Table[g[m], {m, 0, 16}], 1])
```

E_6: For E_6, the highest root (weight of the adjoint representation), 78 is {0,0,0,0,0,1}. Its (78−6)/2 = 36 positive root vectors are found as follows:

Input (3)

```
ClearAll [a, e, g];
```

$$a = \{\{2, \bar{1}, 0, 0, 0, 0\}, \\ \{\bar{1}, 2, \bar{1}, 0, 0, 0\}, \\ \{0, \bar{1}, 2, \bar{1}, 0, \bar{1}\}, \\ \{0, 0, \bar{1}, 2, \bar{1}, 0\}, \\ \{0, 0, 0, \bar{1}, 2, 0\}, \\ \{0, 0, \bar{1}, 0, 0, 2\}\};$$

```
g[0] = {{0, 0, 0, 0, 0, 1}};
g[1] = Table[ Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[p]]]],
{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}];
e[x_] : =  e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]]  == 1, g[x][[j]]- a[[i]],
If [g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]], {i, 6}], {j, Length[g[x]]}],
Table[Table[ If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, 6}],
{l, Length[g[x- 1]]}]}, 2]];
g[x_] : =  g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];
Flatten[Table[g[m], {m, 0, 10}], 1])
```

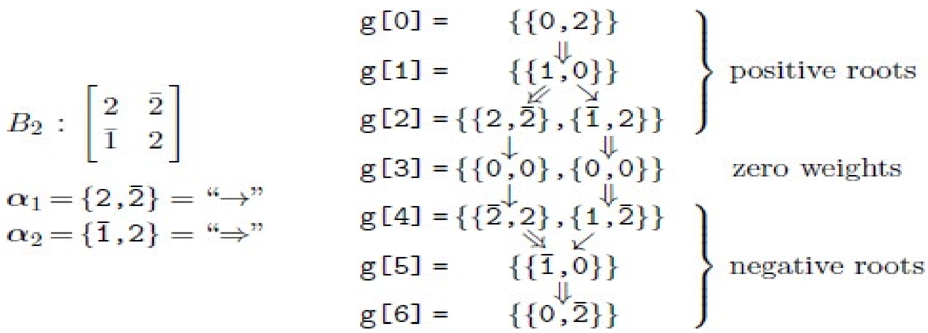
For the infinite sequences of Lie algebras A_n, B_n, C_n, D_n, we recall the low-dimensional isomorphisms [10]

$$C_1 \approx B_1 \approx A_1, \qquad C_2 \approx B_2, \qquad D_2 \approx A_1 \oplus A_1, \qquad D_3 \approx A_3. \tag{3.4}$$

For this reason, we provide the Mathematica code below as follows: A_n for n > 1, B_n and C_n for n > 2, D_n for n > 3, and provide the two remaining (low-n) cases explicitly, for illustration purposes:

$$A_1: \underbrace{[2]}_{\text{Cartan matrix}}, \quad \underbrace{g[0]=\{\{2\}\}}_{\text{positive root}}, \quad \underbrace{g[1]=\{\{0\}\}}_{\text{zero weight}}, \quad \underbrace{g[2]=\{\{\bar{2}\}\}}_{\text{negative root}}, \tag{3.5}$$

which correspond to the well-known $\{J_+, J_z, J_-\}$ generators of SU_2.



The Cartan matrix of C_2 is the transpose of that of B_2, so that the positive simple roots of C_2 are the simply the swapped simple roots of B_2, whereby the root system of C_2 is identical as shown in (3.6).

A_n: For A_n, the dimension of the adjoint representation is n(n+2) and the number of positive root vectors is (n(n+2) − n)/2 = n(n−1)/2. The Mathematica code computing the positive root vectors of A_n, for n = 5 for example, is:

Input (4)

```
ClearAll [n, d, a, g, e];
n = 5; (* n = 2, 3, 4, ... *)
  d = {{2, 1̄, 0},
       {0, 1̄, 2},
       {1̄, 2, 1̄}};
a = If[n > 1, Flatten[{{{PadLeft[d[[1]], n, 0, n- 3]}},
{Table[ PadLeft[ d[[3]], n, 0, n- i- 2], {i, n- 2}]},
{{PadRight[ d[[2]], n, 0, n- 3]}}}, 2], {2}];
g[0] = {RotateLeft[PadRight[{1, 1}, n, 0], 1]};
g[1] = Table[ Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[p]]]],
{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}];
e[x_] : =  e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]]  == 1, g[x][[j]]- a[[i]],
If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]], {i, n}], {j, Length[g[x]]}],
Table[Table[ If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, n}],
{l, Length[g[x- 1]]}]}, 2]];
g[x_] : =  g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];
Flatten[Table[g[m], {m, 0, n-1}], 1])
```

B_n: For B_n, the dimension of the adjoint representation is n(2n+1) and the number of positive root vectors is $(n(2n+1)-n)/2 = n^2$. The Mathematica code computing the positive root vectors of B_n, for n = 5 for example, is:

Input (5)

```
ClearAll [n, d, a, g, e];
n = 5; (* n = 3, 4, 5, ... *)
```

$$d = \{\{2,\ \bar{1},\ 0\},\ \{\bar{1},\ 2,\ \bar{2}\},\ \{0,\ \bar{1},\ 2\},\ \{\bar{1},\ 2,\ \bar{1}\}\};$$

```
a = Flatten[{{{PadLeft[d[[1]], n, 0, n- 3]}},
 {Table[ PadLeft[d[[4]], n, 0, n- i- 2], {i, n- 3}]},
{{PadRight[ d[[2]], n, 0, n- 3]}}, {{PadRight[d[[3]], n, 0, n- 3]}}}, 2];
g[0] = {PadRight[{0, 1, 0}, n, 0]};
g[1] = {Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[1]]]]};
e[x_] : = e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]] == 1, g[x][[j]]- a[[i]],
 If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]]], {i, n}], {j, Length[g[x]]}],
 Table[Table[ If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, n}],
 {l, Length[g[x- 1]]}]}, 2]];
g[x_] : =  g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];
Flatten[Table[g[m], {m, 0, 2n-2}], 1])
```

C_n: Similarly to B_n, the dimension of the adjoint representation of C_n is also n(2n+1) and the number of positive root vectors is also $(n(2n+1)-n)/2 = n^2$. The Mathematica code computing the positive root vectors of C_n, for n = 5 for example, is:

Input (6)

```
ClearAll [n, d, a, g, e];
n = 5; (* n = 3, 4, 5, ... *)
```

$$d = \{\{2,\ \bar{1},\ 0\},\ \{\bar{1},\ 2,\ \bar{2}\},\ \{0,\ \bar{1},\ 2\},\ \{\bar{1},\ 2,\ \bar{1}\}\};$$

```
a = Transpose[Flatten[{{{PadLeft[d[[1]], n, 0, n- 3]}},
{Table[PadLeft[d[[4]], n, 0,n- i- 2], {i, n- 3}]},
 {{PadRight[d[[2]], n, 0, n- 3]}}, {{PadRight[d[[3]], n, 0, n- 3]}}}, 2]];
g[0] = {PadRight[{2}, n, 0]};
g[1] = {Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 2]][[1]]]]};
e[x_] : =  e[x] = Union[Flatten[{Table[Table[If[g[x][[j]][[i]] == 1, g[x][[j]]- a[[i]],
If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]]], {i, n}], {j, Length[g[x]]}],
Table[Table[If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, n}],
{l, Length[g[x- 1]]}]}, 2]];
g[x_] : =  g[x] = Delete[e[x- 1], 1];
Flatten[Table[g[m], {m, 0, 2n-2}], 1]
```

D_n: For D_n the dimension of the adjoint representation is n(2n−1) and the number of positive root vectors is $(n(2n-1)-n)/2 = n(n-1)$. The Mathematica code computing the positive root vectors of D_n, for n = 5 for example, is:

Input (7)

```
ClearAll [n, d, a, g, e];
n = 5; (* n = 4, 5, 6, ... *)
```

```
d = {{2, 1̄, 0, 0},
     {1̄, 2, 1̄, 1̄},
     {0, 1̄, 2, 0},
     {0, 1̄, 0, 2},
     {1̄, 2, 1̄, 0}};

a = Flatten[{{{PadLeft[d[[1]], n, 0, n- 4]}},
{Table[PadLeft[d[[5]], n, 0, n- i- 3], {i, n- 4}]},
{{PadRight[d[[2]], n, 0, n- 4]}}, {{PadRight[d[[3]], n, 0, n- 4]}},
{{PadRight[d[[4]], n, 0, n- 4]}}}, 2];
g[0] = {PadRight[{0, 1, 0}, n, 0]};
g[1] = Table[ Flatten[g[0]]- a[[Flatten[Position[Flatten[g[0]], 1]][[p]]]],
{p, Length[Flatten[Position[Flatten[g[0]], 1]]]}];
e[x_] : = e[x] = Union[Flatten[{Table[ Table[If[g[x][[j]][[i]] == 1, g[x][[j]]- a[[i]],
 If[g[x][[j]][[i]] == 2, g[x][[j]]- a[[i]]]], {i, n}], {j, Length[g[x]]}],
 Table[Table[ If[g[x- 1][[l]][[k]] == 2, g[x- 1][[l]]- 2 a[[k]]], {k, n}],
 {l, Length[g[x- 1]]}]}, 2]];
g[x_] : = g[x] = If[MemberQ[e[x- 1], Null], Delete[e[x- 1], 1], e[x- 1]];
Flatten[Table[g[m], {m, 0, 2n-4}], 1])
```

As with the E_8 code Input (1), replacing

$$\text{Flatten [Table [g[m],\{m,0,m}_{max}\}],1] \text{ Do [Print [g[m]], \{m,0,2m}_{max}+2\}] \tag{3.7}$$

In the last line of the codes Input (2)–(7), where m_{max} is the index limit as shown above, prints all the roots at their actual level, forming the characteristic spindle-shaped listing.

The highest root, the level of positive simple root vectors (i.e., the height of the tower of positive roots) and the dimension of the adjoint representation can be found in Table 8 of [11], while Table 9 of Ref [11] gives the positive root systems of a few low-rank simple Lie groups. We leave it to the diligent Reader to adapt the above Mathematica codes for the remaining simple Lie groups, G_2 and F_4.

— * —

In constructing $T^6/\mathbb{Z}_N = (R^6/\Lambda)/\mathbb{Z}_N$ orbifolds for superstring theory and its M-theory extension, the choices of the $\mathbb{Z}_N$ shift vectors (representing the embedding in the gauge group) are restricted. For example, in M-theory, the shift vectors must satisfy a supersymmetry condition, while in string theory they satisfy an additional modular invariance condition; herein, we impose only the former.

We give an example of $\mathbb{Z}_7$ vectors. There are 428 eight-component vectors that may be constructed with the components taking values in the standard range. The supersymmetry restriction requires that the components of a $\mathbb{Z}_N$ vector add up to an integer [14]. The following code produces all such "supersymmetric" $\mathbb{Z}_7$-vectors. We have shown only a sample of the output. Note that in order to find all the possible vectors preserving supersymmetry, we need to consider all permutations of the components of each one of the vectors produced by this code; this is accomplished by applying the Mathematica function Permutations [list] to each $\mathbb{Z}_7$-vector produced in Output (8), below.

The code under Input (8) proceeds as follows:

a: stores a list of standard (fractional) nonzero values for the components of the $\mathbb{Z}_7$-vectors (2.1). For general $\mathbb{Z}_N$, replace the values with proper fractions $\frac{k}{N}$, for $k = 1,\ldots,N$.

b: stores, for $2 \le i \le 7$, a list of *i*-tuples of possibly repeated component-values from a, sorted and with duplicate *i*-tuples removed. For a Lie group of rank r, let $2 \le i \le (r-1)$.

def.: The list-function complete [*li*st] appends the negative of the total sum of the list-components, reduced mod 1, i.e., it appends a (possibly 0) component that makes the total sum into an integer.

c: applies the list-function "complete[*list*]" throughout the list of *i*-tuples "b", completing them into *i*-tuples with integral totals.

q: stores the i-tuples from " c," padded by zeros to form 8-vectors, with sorted components, removed duplicates and sorted as vectors. For a Lie group of rank *r*, replace $PadRight\,[c\,[[i]],\,8] \to PadRight\,[c[[i]],\,r]$.

To relax the supersymmetric condition for the total sum of the components of the N-vectors v to be integral, omit line "c," and replace c ⟶ b in line "q"; the line defining the list-function complete [list] thus becomes unused and may also be omitted.

Input (8)

```
ClearAll[a, b, c, q]; (* Clear arrays from previous computations *)

a = {1/7,2/7,3/7,4/7,5/7,6/7};

b = Union[Sort/@ Flatten[Table[Tuples[a,i],{i,2,7}],1]];

complete[list_] : =  Append[list, Mod[-Total[list], 1]];

c = complete/@ b;

q = Sort[Union[Sort /@ Table[PadRight[c[[i]], 8], {i, 1, Length[c]}]]];

"Total no. of Z7 Vectors"

Length[q]
```

Output (8)

$$\Big\{\Big\{0,0,0,0,0,0,\tfrac{1}{7},\tfrac{6}{7}\Big\},\quad \Big\{0,0,0,0,0,0,\tfrac{2}{7},\tfrac{5}{7}\Big\},\quad \Big\{0,0,0,0,0,0,\tfrac{3}{7},\tfrac{4}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{1}{7},\tfrac{1}{7},\tfrac{5}{7}\Big\},$$

$$\Big\{0,0,0,0,0,\tfrac{1}{7},\tfrac{2}{7},\tfrac{4}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{1}{7},\tfrac{3}{7},\tfrac{3}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{2}{7},\tfrac{2}{7},\tfrac{3}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{2}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},$$

$$\Big\{0,0,0,0,\tfrac{3}{7},\tfrac{5}{7},\tfrac{6}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{4}{7},\tfrac{4}{7},\tfrac{6}{7}\Big\},\quad \Big\{0,0,0,0,0,\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7}\Big\},\quad \ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots,\quad \Big\{\tfrac{3}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{3}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{3}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7},,\tfrac{5}{7},\tfrac{5}{7}\Big\},$$

$$\Big\{\tfrac{3}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{3}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7}\Big\},$$

$$\Big\{\tfrac{4}{7},\tfrac{4}{7},\tfrac{4}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{4}{7},\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{4}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{6}{7},\tfrac{6}{7},\tfrac{6}{7}\Big\},\quad \Big\{\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7},\tfrac{5}{7}\Big\}\Big\}$$

"Total No. of Z7 Vectors" 428

One may use a similar code for generating general $\mathbb{Z}_N$ shift vectors in the root lattice for N≠7.

SUBGROUPS OF G

Our next step is to find all the regular, maximal-rank subgroups of G, using (2.1)–(2.2).

Our task is indeed closely related to the well-known problem of finding the regular subalgebras of the Lie algebra of G, which is accomplished by using the extended Dynkin diagram technique [6]; see also Refs [10-12]. The procedure starts with removing in every possible way one node from the extended Dynkin diagram of the Lie algebra of the original group G, producing a collection of Dynkin diagrams of the first list of maximal regular subalgebras. One then iterates this procedure for every Lie algebra from this first list. While this procedure is not perfect, all of the very few required corrections are known by now [12]

Many of the subalgebras are also found by the quicker method of removing from the extended Dynkin diagram of the group G several nodes in all possible ways at once, and reading off the subalgebra represented by the remainder. For example **(Figure 2),** if we take out the nodes α_1, α_6 and α_7 from the extended Dynkin diagram of E_8, we get $D_5 + A_1$; see **Figure 2.** Notably, however, this does not produce all subalgebras, such as for example $D_4 + D_4 \subset E_8$, which is obtained by the above-outlined iterative method, as shown in **Figure 3.** The resulting complete list of regular subalgebras of E_8 has been known since Ref [6].

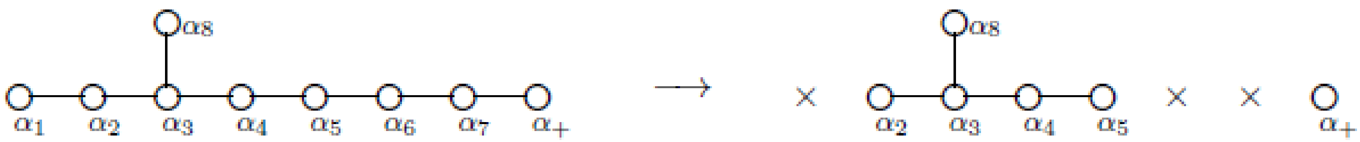


Figure 2. Removing the nodes α_1, α_6 and α_7 from the extended Dynkin diagram of E_8 gives the regular subalgebra D_5+A_1. "α_+" denotes the extending node; " × " denote the locations of the removed nodes.

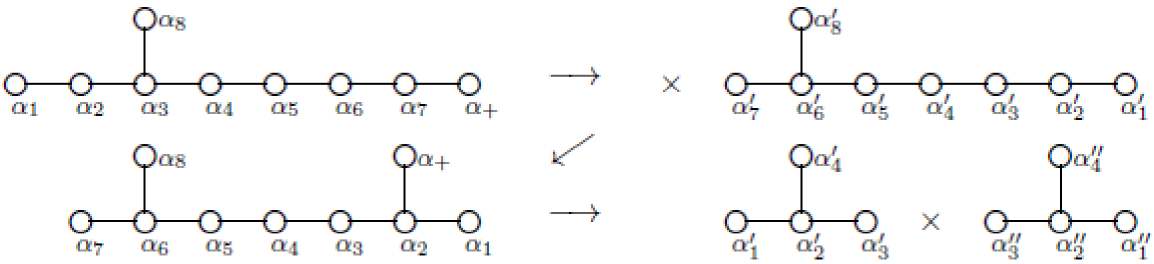


Figure 3. Removing the node α ' from the extended Dynkin diagram of E_8 (top left) gives the maximal regular subalgebra D_8 (top right). Removing the node α_4 from the extended Dynkin diagram of D_8 (bottom left) gives the regular subalgebra $2D_4 \subset D_8 \subset E_8$ (bottom right).

We then pass to the corresponding compact Lie (sub)groups. Rather importantly, the $\mathbb{Z}_N$-invariant regular subgroups are necessarily of maximal rank and include rank$(G)-(H_I)$ abelian factors U_1, where H_I is the semisimple factor of the $\mathbb{Z}_N$-invariant regular subgroup $H_I \subset G$. This fact renders the centralizer of the semisimple factor in each $\mathbb{Z}_N$-invariant subgroup trivial, and effectively removes the distinction between inequivalently embedded subgroups; see Appendix for details. The resulting list of maximal-rank regular subgroups of E_8 is given in **Table 1.**

Table 1. Regular subgroups of E_8 and their identifiers described in the text. Double daggers (‡) indicates subgroups that have two inequivalent embeddings in E_8 [6]. Whereas these are not distinguished by the identification of $\mathbb{Z}_N$ -invariant subgroups described herein, their distinction is also made irrelevant by the inclusion of the $U(1)^{8-r}$ factors and by focusing exclusively on the adjoint representation; see Appendix.

	Subgroups	p	m	r	c		Subgroups	p	m	r	c
0	E_8	120	0	8		36	$SU_7 \times U_1^2$	21	0	6	✓
1	SO_{16}	56	0	8		37	$SO_{10} \times SU_2 \times U_1^2$	21	1	6	✓
2	SU_9	36	0	8	✓	38	$[SU_6]^‡ \times SU_2 \times U_1^2$	16	1	6	✓
3	$SU_8 \times SU_2$	29	1	8		39	$SO_8 \times SU_3 \times U_1^2$	15	0	6	✓
4	$SU_6 \times SU_3 \times SU_2$	19	1	8		40	$SO_8 \times SU_2^2 \times U_1^2$	14	2	6	✓
5	SU_5^2	20	0	8		41	$SU_5 \times SU_3 \times U_1^2$	13	0	6	
6	$SO_{10} \times SU_4$	26	0	8		42	$[SU4^2]‡ \times U_1^2$	12	0	6	
7	$E_6 \times SU_3$	39	0	8		43	$SU_5 \times SU_2^2 \times U_1^2$	12	2	6	
8	$E_7 \times SU_2$	64	1	8		44	$SU_4 \times SU_3 \times SU_2 \times U_1^2$	10	1	6	
9	$SO_1^2 \times SU_2^2$	32	2	8		45	$SU3^3 \times U_1^2$	9	0	6	
10	$SO_8 \times SU_2^4$	16	4	8	✓	46	$SU_4 \times SU_2^3 \times U_1^2$	9	3	6	
11	$SU2^8$	8	8	8		47	$SU_3^2 \times SU_2^2 \times U_1^2$	8	2	6	
12	$SU4^2 \times SU_2^2$	14	2	8	✓	48	$SU_3 \times SU_2^4 \times U_1^2$	7	4	6	
13	SO_8^2	24	0	8		49	$SU_2^6 \times U_1^2$	6	6	6	
14	SU_3^4	12	0	8		50	$SO_{10} \times U_1^3$	20	0	5	
15	$E_7 \times U_1$	63	0	7	✓	51	$SU_6 \times U_1^3$	15	0	5	✓
16	$SO_{14} \times U_1$	42	0	7	✓	52	$SO_8 \times SU_2 \times U_1^3$	13	1	5	
17	$E_6 \times SU_2 \times U_1$	37	1	7	✓	53	$SU_5 \times SU_2 \times U_1^3$	11	1	5	
18	$SO_{12} \times SU_2 \times U_1$	31	1	7		54	$SU_4 \times SU_3 \times U_1^3$	9	0	5	
19	$[SU_8]^‡ \times U_1$	28	0	7	✓	55	$[SU_4 \times SU_2^2]‡ \times U_1^3$	8	2	5	
20	$SO_{10} \times SU_3 \times U_1$	23	0	7	✓	56	$SU_3^2 \times SU_2 \times U_1^3$	7	1	5	
21	$SO_{10} \times SU_2^2 \times U_1$	22	2	7	✓	57	$SU_3 \times SU_2^3 \times U_1^3$	6	3	5	
22	$SU_7 \times SU_2 \times U_1$	22	1	7	✓	58	$SU_2^5 \times U_1^3$	5	5	5	
23	$SO_8 \times SU_4 \times U_1$	18	0	7		59	$SO_8 \times U_1^4$	12	0	4	
24	$SU_6 \times SU_3 \times U_1$	18	0	7		60	$SU_5 \times U_1^4$	10	0	4	
25	$SU_6 \times SU_2^2 \times U_1$	17	2	7		61	$SU_4 \times SU_2 \times U_1^4$	7	1	4	
26	$SU_5 \times SU_4 \times U_1$	16	0	7	✓	62	$SU_3^2 \times U_1^4$	6	0	4	
27	$SO_8 \times SU_2^3 \times U_1$	15	3	7	✓	63	$SU_3 \times SU_2^2 \times U_1^4$	5	2	4	
28	$SU_5 \times SU_3 \times SU_2 \times U_1$	14	1	7	✓	64	$[SU_2^4]‡ \times U_1^4$	4	4	4	
29	$SU_4^2 \times SU_2 \times U_1$	13	1	7		65	$SU_4 \times U_1^5$	6	0	3	
30	$SU_4 \times SU_3 \times SU_2^2 \times U_1$	11	2	7		66	$SU_3 \times SU_2 \times U_1^5$	4	1	3	
31	$SU_3^3 \times SU_2 \times U_1$	10	1	7		67	$SU_2^3 \times U_1^5$	3	3	3	

32	$SU_4 \times SU_2^4 \times U_1$	10	4	7		68	$SU_3 \times U_1^6$	3	0	2	
33	$SU_2^7 \times U_1$	7	7	7		69	$SU_2^2 \times U_1^6$	2	2	2	
34	$E_6 \times U_1^2$	36	0	6	✓	70	$SU_2 \times U_1^7$	1	1	1	
35	$SO_1^2 \times U_1^2$	30	0	6	✓	71	U_1^8	0	0	0	

For this list of all maximal-rank regular subgroups of E_8, we calculate the number of positive root vectors for each subgroup and list them in column p of Table 1. The values of the other identifiers (r, m and possibly m_2, m_3,...) turned out not to be necessary in most cases for our purposes[5]. Before using them, we found the possible candidates which are $\mathbb{Z}_7$-invariant subgroups of E_8 through a procedure given in Input/output (9). This greatly reduced the complexity of the codes in the next section and saves in the Mathematica evaluation time.

We have 428 $\mathbb{Z}_7$ shift vectors in Output (8) and once we take their permutations, this gives a total of 823, 542 shift vectors. We take the first $\mathbb{Z}_7$ $\{0,0,0,0,0,0,\frac{1}{7},\frac{6}{7}\}$ vector from the previous section and calculate the number of positive root vectors that satisfy the condition $\mathbf{p}\cdot\mathbf{v}\in\mathbb{Z}$ using the following code:

Input (9)

```
p = (not shown here: 120 positive roots of E8 from Output());
v = Flatten[Table[Permutations[q[[i]]], {i, 1, 1}], 1];
u = Table[Table[p[[i]].v[[j]], {i, Length[p]}], {j, Length[v]}];
w = Table[ Table[IntegerQ[u[[j, i]]], {i, Length[p]}], {j, Length[v]}];
r = Table[Count[w[[j]], True], {j, Length[v]}];
Union[r] >>> Z7_Roots;)Opt
```

Output (9)

```
{37,42}
```

The code under Input (9) reads the 428 $\mathbb{Z}_7$-vectors from Output (8) into the list "q" and the 120 positive roots from Output (1) into the list "p" and then proceeds as follows:

v: The list of $\mathbb{Z}_7$-vectors v obtained as permutations of the first vector in Output (8).

u: Stores the dot products between each one of the vectors from "v" and the 120 positive roots in "p".

w: Finds the integral dot products in "u".

r: Counts the number of integral dot products in "u", which is the total number of positive roots (in "p") satisfying the condition $\mathbf{p}\cdot\mathbf{v}\in\mathbb{Z}$, for each one of the vectors v in "v".

The Output of this evaluation ({37,42}) is written in an external file Z7_Roots. We do this evaluation for the other $\mathbb{Z}_7$ vectors in Output (8) and the results are collected from the text file Z7_Roots. This gives the possible values of the identifier p for $\mathbb{Z}_7$ vectors as

P : {14, 15, 16, 21, 22, 23, 28, 30, 36, 37, 42, 63} (4.1)

This narrows down our choices to 21 subgroups of E_8, marked by a check in column c of **Table 1.** Now we use the values of m (number of A_1 factors in $H_l \subset E_8$) to identify the possible subgroups $H_l \subset E_8$. When p and m do not specify $H_l \subset E_8$ unambiguously, we use the values of r (rank of the semisimple part of H_l). The values of the identifiers *p, m* and *r* are also calculated from the root vectors that are invariant (2.1) with respect to a $\mathbb{Z}_7$ shift. This is shown in the next section.

$\mathbb{Z}_N$ INVARIANT SUBGROUPS OF G

To illustrate the procedure of calculating the values of m and r from the $\mathbb{Z}_7$-invariant root vectors we give the same example as in Section 3 of our previous paper [5]. Take the shift vector $\nu=\left\{\frac{1}{7},\frac{1}{7},0,\frac{2}{7},0,0,\frac{3}{7},0\right\}$, which is one of the permutations of $\left\{0,0,0,0,\frac{1}{7},\frac{1}{7},\frac{2}{7},\frac{3}{7}\right\}$. The $\mathbb{Z}_N$-invariant E_8 root vectors are

[5]The identifiers r and m are shown in **Table 1** for completeness and for the benefit of possible generalizations to $\mathbb{Z}_N$-actions where the supersymmetry condition is relaxed. The additional identifiers, m_i in Step 4, are easily added

$$\begin{array}{lll} \{0,0,0,0,1,\bar{1},0,0\}, & \{1,\bar{1},0,0,0,0,0,1\}, & \{1,\bar{1},1,0,0,0,0,\bar{1}\}, \\ \{\bar{1},1,0,0,\bar{1},1,0,0\}, & \{0,0,\bar{1},0,1,0,0,1\}, & \{0,0,0,0,1,0,0,\bar{1}\}, \\ \{1,\bar{1},0,0,0,1,0,0\}, & \{0,0,0,0,0,\bar{1},0,1\}, & \{0,0,1,0,0,\bar{1},0,\bar{1}\}, \\ \{0,1,\bar{1},1,\bar{1},1,\bar{1},0\}, & \{\bar{1},0,1,\bar{1},0,0,1,0\}, & \{1,\bar{1},1,0,\bar{1},0,0,0\}, \\ \{\bar{1},1,\bar{1},0,1,\bar{1},0,1\}, & \{\bar{1},1,0,0,1,\bar{1},0,\bar{1}\}, & \{0,0,\bar{1},0,0,0,0,2\} \end{array} \tag{5.1}$$

and are thus invariant under the action of the group $\mathbb{Z}_7$ generated by this shift. Call these root vectors t[i], i = 1, 2, ...15, and set p = 15.

Next, we need to identify which subgroup $H_I \subset E_8$-from among those listed in **Table 1**-do these roots (together with their negatives and the Cartan root vectors) generate. We look for possible relations in the form t[i] + t[j] = t[k] and find the following:

$$t[2]+t[14]=t[1], \quad t[3]+t[13]=t[1], \quad t[5]+t[9]=t[1], \quad t[6]+t[8]=t[1], \tag{5.2a}$$

$$t[3]+t[15]=t[2], \quad t[5]+t[12]=t[2], \quad t[7]+t[8]=t[2], \tag{5.2b}$$

$$\left.\begin{array}{l} t[6]+t[12]=t[3] \\ t[7]+t[9]=t[3] \end{array}\right\} \quad \left.\begin{array}{l} t[6]+t[15]=t[5] \\ t[7]+t[13]=t[5] \end{array}\right\} \quad \left.\begin{array}{l} t[9]+t[15]=t[8] \\ t[12]+t[13]=t[8] \end{array}\right\} \tag{5.2c}$$

$$t[7]+t[14]=t[6], \quad t[12]+t[14]=t[9], \quad t[14]+t[15]=t[13], \tag{5.2d}$$

$$t[10]+t[11]=t[4]. \tag{5.2e}$$

Since the root vectors t[7], t[10], t[11], t[12], t[14] and t[15] cannot be expressed as a sum of any other root vectors, they must correspond to 6 positive, simple root vectors in H_I. The rank of the semisimple part of H_I then must be 6, and the remaining two zero weights correspond to a $U(1)^2$ factor. Also, all the 15 root vectors appear in (5.2), meaning this H_I has no A_1 factors, each of which would have had to have a single, isolated, positive root vector. From these $\mathbb{Z}_7$-invariant root vectors the variables m and rare defined as:

m is the number of root vectors that do not appear in the equation of the form t[i] + t[j] = t[k] and so must be single, isolated, positive root vectors; here, m = 0.

r is the number of root vectors that do not appear on the right side of the relations of the form t[i] + t[j] = t[k] and so must be simple; here, r = 6.

Using {p, m, r} = {15, 0, 6}, we identify unambiguously the subgroup from **Table 1**, as $SO_8 \times SU_3$. Observe that this is indeed consistent with the structure of the relations (5.2):

The positive roots t[4], t[10] and t[11] form a separate rank-2 positive root system where t[10] and t[11] are simple and t[4] is their sum (5.2e); this can correspond only to SU_3.

The positive roots t[6], t[9] and t[13] are each obtained as a sum (5.2d) of two of the positive simple roots {t[7], t[12], t[14], t[15]}, and so must be one level above these positive simple roots.

Expressing t[6], t[9] and t[13] in this way, t[3], t[5] and t[8] are each found to be a sum (5.2c) of three of the positive simple roots, and so are two levels above the positive simple roots.

In this way, t[2] = t[7] + t[12] + t[14] + t[15] is a sum (5.2b) of all four distinct positive simple roots, while t[1] = t[2] + t[14] = t[7] + t[12] + 2 t[14] + t[15] has one more positive simple root (5.2a). Therefore, t[2] and t[1] occupy respectively the third and fourth level above the positive simple roots.

These facts are consistent with {t[7], t[12], t[14], t[15]; t[6], t[9], t[13]; t[3], t[5], t[8]; t[2]; t[1]} forming the positive root system of SO(8), i.e., its Lie algebra D_4. As it turns out, such a more detailed study was not needed in determining the list of $\mathbb{Z}_7$-invariant subgroups of E_8 in **Table 2** and the identifiers {p, m, r} did suffice to this end.

Table 2. $\mathbb{Z}_7$-invariant subroups of E_8

	Group		Group		Group		Group
1	E_7	5	SO_{12}	9	SU_8	13	$SU_5 \times SU_4$
2	$E_6 \times SU_2$	6	$SO_{10} \times SU_3$	10	$SU_7 \times SU_2$	14	$SU_5 \times SU_3 \times SU_2$
3	E_6	7	$SO_{10} \times SU_2$	11	SU_7		
4	SO_{14}	8	$SO_8 \times SU_3$	12	$SU_6 \times SU_2$		

We employ this analysis in the construction of the Mathematica codes below and using the identifiers {p, m, r} identify the fourteen subgroups of E_8 that are invariant under a $\mathbb{Z}_7$ shift listed in **Table 2,** and so in fact the complete $\mathbb{Z}_7$ group action generated by that shift.

Input (10)

```
q = (not shown here: 428 ℤ7 vectors from Output (8));
CleanSlate[]:
v = Flatten[Table[Permutations[q[[i]]], {i, 1, 1}], 1];
u = Table[Table[p[[i]].v[[j]], {i, Length[p]}], {j, Length[v]}];
w = Table[ Table[IntegerQ[u[[j, i]]], {i, Length[p]}], {j, Length[v]}];
s = Table[Flatten[Position[w[[k]], True]], {k, Length[w]}];
t = Table[ Table[p[[s[[j]][[i]]]], {i, Length[s[[j]]]}], {j, Length[w]}];
Φ [k_] : =  Φ [k] = Evaluate[ b = Table[ Table[t[[k]][[i]] + t[[k]][[j]],
{i, Length[t[[k]]]}], {j, Length[t[[k]]]}];
c = Table[ Table[MemberQ[t[[k]], b[[i, j]]], {i, Length[t[[k]]]}], {j, Length[t[[k]]]}];
f = Position[c, True];
g = Union[Table[Sort[f[[i]]], {i, Length[f]}]];
x = Table[g[[i]][[1]], {i, Length[g]}];
y = Table[g[[i]][[2]], {i, Length[g]}];
h = Table[t[[k]][[x[[i]]]] + t[[k]][[y[[i]]]], {i, Length[x]}];
z = Flatten[Table[Position[t[[k]], h[[i]]], {i, Length[h]}]];
o = Table[l, {l, Length[t[[k]]]}];
m = Length[Complement[o, Union[x, y, z]]];
r = Length[Complement[o, z]];];
Table[If[Length[t[[k]]] == 14, Evaluate[ Φ [k]; If[m == 2, If[r == 6, a[1] a[1] d[4],
 If[r  == 8, a[1] a[1] a[3] a[3]]], a[1] a[2] a[4]]],
If[Length[t[[k]]] == 15, Evaluate[ Φ [k]; If[m == 0, If[r  == 5, a[5],
If[r == 6, a[2] d[4]]], a[1] a[1] a[1] d[4]]],
If[Length[t[[k]]] == 16, Evaluate[ Φ [k]; If[m == 0, a[3] a[4], If[m == 1, a[1] a[5],
If[m == 4, a[1] a[1] a[1] a[1] d[4]]]]],
If[Length[t[[k]]] == 21, Evaluate[ Φ [k]; If[m == 0, a[6], a[1] d[5]]],
If[Length[t[[k]]] == 22, Evaluate[ Φ [k]; If[m == 1, a[1] a[6], a[1] a[1] d[5]]],
If[Length[t[[k]]] == 23, a[2] d[5],
If[Length[t[[k]]] == 28, a[7],
If[Length[t[[k]]] == 30, d[6],
If[Length[t[[k]]] == 36, Evaluate[ Φ [k]; If[r  ==  6, e[6], a[8]]],
If[Length[t[[k]]] == 37, a[1] e[6],
If[Length[t[[k]]] == 42, d[7],
If[Length[t[[k]]] == 63, e[7]]]]]]]]]]]]], {k, Length[t]}];
Union[%]>>>Z7_Groups;
```

Output (10)

{d [7], a[1]e[6]}

The code under Input (10) reads the 428 $\mathbb{Z}_7$-vectors from Output (8) into the list “q” and the 120 positive roots from Output (1) into the list “p” and then proceeds as follows:

v, u, w: Have the same meaning as in Input (9).

s, t: For each one of the $\mathbb{Z}_7$ shift-vectors in “v”, these find the set of positive root vectors of E_8 that have integral scalar products with the $\mathbb{Z}_7$ shift-vector.

Φ: This function uses the analysis as stated after equation (5.1) to find the values of m and r as defined in section 2. Note that Φ is a function which uses the variables “c”, “f”, “g”, “x”, “y”, “h”, “z” and “o” to evaluate “m” and “r” (which have the meaning of m and r from Section 2). This function is evaluated only when the number of $\mathbb{Z}_7$-invariant roots of E_8 (in the code this number is Length [t[[k]]]) is not enough to identify the subgroup H_l as discussed in this section. The quantity Length [t[[k]]] is the number of $\mathbb{Z}_7$-invariant root vectors (= p), “m” is the number of A_1 factors (= m) and “r” is the rank (r) of a group. These three variables are calculated from the $\mathbb{Z}_7$-invariant root vectors as explained in the above example, the $E_8 \supset SO(8)\times SU(3)$ subgroup. The output of this evaluation is written in an external file $\mathbb{Z}_7$ Groups where a group A_n is identified as a[n], D_n as d[n] and E_n as e[n].

For other orbifolds there are situations where p, m and r do not suffice to specify the group unambiguously. In those cases we look for $A_2, A_3, \cdots$ factors in H_l by looking at root vector relations. For example, an A_2 factor would have to be spanned by three root vectors {t[i], t[j], t[k]} that satisfy an equation of the form t[i] + t[j] = t[k] and occur in no equation involving any other root vectors. Equivalently, we can look for root vectors that do not appear in any equation of the form t[i] + t[j] + t[k] = t[l].

AUTOMATION

Due to limitations of computer’s processor speed and memory, it may be necessary to partition the computation. The following shows how it may be done for the $\mathbb{Z}_7$ orbifold example in M-theory.

Collect all the $\mathbb{Z}_7$ vectors q in Output (8) and all the E_8 root vectors p in Output (1) of section 3 and put them in a notebook, say, NB_0. Use the package ‘CleanSlate’[6][14] and put this in one of Mathematica’s home directory ($HomeDirectory). This package helps in clearing the Mathematica kernel memory so that successive evaluations can use the maximum possible memory. The input of NB_0 are as follows:

Input (11)

q = ; (no output shown here: 428 $\mathbb{Z}_7$ vectors from Output (8))

p = ; (no output shown here: 120 positive root vectors of E_8 from Output (1))

<< CleanSlate.m;

orbifold = EvaluationNotebook[];

NotebookSave [orbifold]

NotebookOpen ["NB_1.nb"])

(ii) We create a notebook Z7_Generic in $HomeDirectory which contains the code of Input (1) with some added lines of codes to make use of the automation process:

Input (12)

NotebookClose [orbifold]

CleanSlate[];

v = Flatten [Table[Permutations[q[[i]]], {i, α, α}], 1];

u = Table [Table[p[[i]].v[[j]], {i, Length[p]}], {j, Length[v]}];

w = Table [Table[IntegerQ[u[[j, i]]], {i, Length[p]}], {j, Length[v]}]; r = Table[Count[w[[j]], True], {j, Length[v]}];

Union[r] >>> Z7_Roots;

orbifold = EvaluationNotebook[]; NotebookSave[orbifold]

γ = α + 1;

"NB " <> ToString[γ] <> ".nb";

InputForm [%]

NotebookOpen [%];

Next we create a notebook Z7_Generator with the following set of codes,

Input (13)

Do[NotebookPut[NotebookGet[First[Notebooks["Z7_Generic.nb"]]]/."α"->β];

[6]This package is available on-line at: http://library.wolfram.com/infocenter/MathSource/4718/.

NotebookSave[SelectedNotebook[],"NB "<>ToString[β]<>".nb"];

Pause [2]; NotebookClose [SelectedNotebook[]],β,1,428]

Once the Input (13) is run, it creates 428 notebooks with the contents of Input (12) where the value of

α = 1, 2, 3, ⋯, 428, respectively, for each notebook. The files are created in the $HomeDirectory.

(iii) Our next step is to evaluate these 428 notebooks in a way such that when we open NB_0, it automatically evaluates it's content and the contents of notebooks NB_1, NB_2 and so on so forth. The NotebookClose[orbifold] input line closes the previous notebook that has been evaluated. In this way the screen is not cluttered with open Mathematica notebooks, improving the performance of the computer's memory. The memory is also managed by the input line CleanSlate[]. Note that the 'CleanSlate' package is called in after Mathematica stores the values of q and win its memory which is necessary for the whole evaluation process. The end result is collected from the text file Z7_Roots created in $HomeDirectory and is given in Eq. (4.1).

(iv) We apply a similar procedure for the evaluation of the $\mathbb{Z}_7$

invariant groups, Input (10). The results are collected from the text file Z7_Groups and are summerized in **Table 2.**

In order for the automation process to work we need to make the following changes to Mathematica preferences,

1. Notebook Options → File Options → Notebook Autosave (False → True)
2. Notebook Options → File Options → ClosingAutosave (False → True)
3. Notebook Options → File Options → AutogeneratedPackage (Manual → None)
4. Notebook Options → Evaluation Options → Initialization CellEvaluation (Automatic →True)
5. Notebook Options → Evaluation Options → Initialization CellWarning (True → False)
6. Cell Options → Evaluation Options → Initialization Cell (False →True)

This automation process was first tested and used in version 5.2 of Mathematica, where it worked as designed. For later versions, there appears to be a problem which prevents the evaluation of a notebook when it is opened by another notebook, even though the Initialization CellEvaluation and Initialization.

Cell are changed to True (globally). In those versions of Mathematica, the automation process (iii) can be performed using a code such as:

Input (14)

nb = NotebookOpen ["notebook.nb"];

SelectionMove [nb, All, Notebook];

SelectionEvaluate [nb];

orbifold = EvaluationNotebook [];

NotebookSave [orbifold];

NotebookClose [orbifold];

Corresponding changes need to be made also in Input (11) and Input (12) for this automation process to work.

CONCLUSION

We have shown in detail how to find the $\mathbb{Z}_7$-invariant subgroups of E_8 using Mathematica. These groups, obtained in orbifold M-theory, turn out to be closely related to string theory compactification down to four dimensions: In the limit $x^{11} \rightarrow 0$, the two $\mathbb{Z}_7$-invariant subgroups of E_8 (one on each of the two boundaries of x^{11}) coalesce into $H_{I,L} \times H_{I',R}$ which turn out to coincide with the gauge groups found in $\mathbb{Z}_7$-orbifold models in string theory [14]. We have tested our codes also for $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ orbifolds. The so-obtained subgroups upon the limit $x^{11} \rightarrow 0$ coincide with those found in string theory compactification. This would imply that our codes can also be used for $\mathbb{Z}_8$ and $\mathbb{Z}_{12}$ orbifolds.

In the presence of gauge background fields (Wilson lines) the four-dimensional gauge group breaks down to some smaller groups. Since these Wilson lines provide additional shifts in the group lattice, it should be possible to employ our procedure also in those types of models.

For the simple Lie groups A_n, B_n, C_n, D_n, E_6 and E_7, our procedure can be applied in finding the unbroken gauge symmetry under any $\mathbb{Z}_N$ shifts. In section 3, we provided the root vectors for these groups. As semisimple Lie groups are products of simple Lie groups, the procedure merely needs to be applied to each factor separately.

Finally, our present goal was the demonstration that Mathematica can be used to compute Δ-invariant subgroups of semisimple Lie groups. Having been motivated by applications in M-theory, we have restricted ourselves to "supersymmetric" Δ-actions and to $\Delta = \mathbb{Z}_N$ for simplicity. Generalizations in both respect seem to be worthwhile, but are beyond our present scope. Similarly, it would seem desirable to re-structure and package the computations presented herein into a single, interactive Mathematica package, but that too is beyond our present scope.

APPENDIX

Regular Subalgebras of E_8

Physics applications in grand-unified model building [11] and string-theory and its M-theory extension [1,2] focus on compact classical Lie groups, and often also on an application-dependently restricted subset of their lowest-dimensional unitary representations. Such is the case in [1,5,14], where (1) only the adjoint representation of E_8 is considered, and (2) only the $\mathbb{Z}_N$-invariant subgroups H_l. In particular, the $\mathbb{Z}_N$-invariant subgroups $H_l \subset G$ all satisfy (2.1)–(2.2) and have a vanishing centralizer; see below. Also, finite factors and the real forms of the Lie groups are not considered and we easily pass from Lie algebras to the corresponding compact Lie groups.

A.1. Subalgebras

An exhaustive procedure for listing the regular subalgebras of Lie algebras was provided originally by EB Dynkin [6], is well described in texts [8,10,12], review literature [11] and also in research articles such as Ref [14]. One starts with listing the maximal semisimple regular subalgebras by removing one node from the extended Dynkin diagram of the original algebra. For E_8, these are [6]:

$$E_8 \supset D_8,\ A_8,\ A_7 + A_1,\ A_5 + A_2 + A_1,\ 2A_4,\ D_5 + A_3,\ E_6 + A_2 \text{ and } E_7 + A_1. \tag{A.1}$$

Next, proceed by listing the maximal semisimple regular subalgebras of (A.1), and continue so iteratively. This adds

$$D_6 + 2A_1,\ D_4 + 4A_1,\ 8A_1,\ 2A_3 + 2A_1,\ 2D_4 \text{ and } 4A_2 \tag{A.2}$$

to the list (A.1), completing the list of all semisimple regular subalgebras of maximal rank [6, Table 10]. Non-semisimple maximal subalgebras are now found by applying to the list (A.1)–(A.2) the results in Dynkin's Table 12.a [6]:

$$\begin{array}{l} A_n \supset A_k + A_{n-k-1} + K_1, \quad B_n \supset B_{n-1} + K_1, \quad C_n \supset A_{n-1} + K_1, \\ D_n \supset D_{n-1} + K_1,\ A_{n-1} + K_1, \quad E_6 \supset D_5 + K_1, \quad E_7 \supset E_6 + K_1, \end{array} \tag{A.3}$$

Where $k = 0,1,2,\ldots,n-2$ for n > 1 and $A_0 \overset{\text{def}}{=\!=} \varnothing$, and K_1 is the "null algebra" consisting of a single certan element, generating an abelian factor U(1) in the corresponding Lie groups. For E_8, this produces the listing

$$\begin{array}{c} \left\{\begin{array}{lllllll} A_7', & A_6 + A_1, & A_5 + 2A_1, & A_4 + A_3, & A_4 + A_2 + A_1, & A_3 + A_2 + 2A_1, & A_3 + 4A_1 \\ A_7'', & E_6 + A_1, & D_7, & D_5 + A_2, & D_5 + 2A_1, & D_4 + A_3, & 3A_2 + A_1 \end{array}\right\} + K_1, \\ \{A_4 + 2A_1,\ D_4 + A_2,\ 2A_3,\ 2A_2 + 2A_1,\ A_2 + 4A_1\} + 2K \end{array} \tag{A.4}$$

omitting the non-semisimple subalgebras wherein a K_1 summand is subsumed within a proper A_1 summand in an otherwise identical subalgebra in the listing. The two separate copies of $A_7 + K_1$ however are listed as inequivalent subalgebras, in that $A_7' + K_1 \subset A_8 \subset E_8$ whereas $A_7'' + K_1 \not\subset A_8 \subset E_8$ [6], which is easily traced in the progression from (A.1) to (A.2) to (A.4).

Finally, in addition to the combined listing of 8 + 6 + 19 subalgebras (A.1)–(A.2)–(A.4), the remaining 42 subalgebras are obtained by omitting summands from the entries (A.1)–(A.2)–(A.4) in all possible ways. In doing so, one must take into account that the omitted summands may turn out to be subsumed in the (larger) centralizer in E8, inducing an equivalence of the remaining summand(s). For example, already in the list (A.1) we have the evidently inequivalent rank-8 semisimple subalgebras A_7+A_1 and E_7+A_1.Omitting the larger summands, we obtain two subalgebras $A_1 \subset A_7 + A_1 \subset E_8$ and $A_1 \subset E_7 + A_1 \subset E_8$ .However, it turns out that these two different embeddings are in fact equivalent by E_8-conjugation [6], so that the centralizer of $A_1 \subset E_8$ is always E_7; this E_7-centralizer subsumes the A_7 from the former subalgebra chain.

In turn, omitting A_1 from $A_7 + A_1 \subset E_8$ leaves the rank-7 subalgebra $A_7 \subset A_7 + A_1 \subset E_8$ with A_1 the centralizer in E_8. Since $(A_7 \subset A_7 + A_1) \not\subset A_8 \subset E_8$, the so-obtained subalgebra A_7 cannot be isomorphic to A_7' in (A.4). This identifies $A_7 \subset A_7 + A_1 \not\subset A_8 \subset E_8$ as Dynkin's A_7' in (A.4), since the first subalgebra pattern in A.3 and Dynkin's distinction of A_7' imply that $(A_7 \subset A_7' + K_1) \subset A_8 \subset E_8$.

It turns out that the remaining isomorphic but inequivalently embedded pairs of four subalgebras,

$$A_5 + A_1, \quad 2A_3, \quad A_3 + 2A_1 \quad \text{and} \quad 4A_1, \tag{A.5}$$

are similarly distinguished by their (carefully traced) centralizers in E_8. The resulting 76 proper subgroups corresponding to these algebras (including the $U(1)^{8-r}$ abelian factor corresponding to the Cartan subalgebra) are listed in **Table 1.**

A.2 Maximal-rank regular subgroups

The preservation by the $\mathbb{Z}_N$-actionshift of the abelian factor $U(1)^{8-r}$ in $H_l \subset G$ renders the centralizer of $H_l \subset G$ trivial.

To see this, consider for example the distinct maximal regular subalgebras $A_7' \subset E_8$ and $A_7'' + A_1 \subset E_8$, where $A_{7'} \subset A_8 \subset E_8$ whereas $A_{7''} \not\subset A_8 \subset E_8$ [6].Omitting the A_1 summand from the latter results in two inequivalently embedded A_7 subalgebras of E_8: the centralizer of $A_{7'}$ is 0, while the centralizer of A_7'' is A_1.

Passing to the corresponding compact Lie groups, we thus have the two inequivalently embedded SU_8 subgroups of E_8, shown here paired with their respective centralizers:

$$\{SU_8' \subset E_8, C_{E_8}(SU_8') = U_1\} \quad vs. \quad \{SU_8'' \subset E_8, C_{E_8}(SU_8'') = SU_2\}. \tag{A.6}$$

The $\mathbb{Z}_N$-invariant subgroups (2.1)-(2.2) of E_8 that contains an SU_8 factor is however $SU_8 \times U_1$. In the case of SU_8', this $\mathbb{Z}_N$-invariant U_1 factor is simply all of the centralizere (A.6). For SU_8'' however, the $\mathbb{Z}_N$-invariant U_1 factor is a proper subgroup of the centralizer, $U_1 \subset SU_2$, the centralizer of which is $C_{SU_2}(U_1) = 1$. Therefore, we obtain that

$$\{(SU_8' \times U_1) \subset E_8, C_{E_8}(SU_8' \times U_1) = 1\} \quad vs. \quad \{(SU_8'' \times U_1) \subset E_8, C_{E_8}(SU_8'' \times U_1) = 1\}. \tag{A.7}$$

It then follows that the two $\mathbb{Z}_N$-invariant subgroups $SU_8 \times U_1 \subset E_8$, differing in the inequivalently embedded SU_8 factors, nevertheless have the same (trivial) centralizer in E_8. Finally, we are herein considering neither how the $\mathbb{Z}_N$-variant complement of the adjoint representation transforms with respect to the ^{n}-invariant subgroup H_I nor how any other E_8-representations might transform under H_I. This then leaves no way to distinguish the two copies of $SU_8 \times U_1$, which is why only one copy is listed in **Table 1.**

The situation is similar for the other four subgroups, $SU_6 \times SU_2 \times U_1^2$, $SU_4^2 \times U_1^2$, $SU_4 \times SU_2^2 \times U_1^3$, $SU_2^4 \times U_1^4$, and only one copy of each of these is listed in **Table 1.**

It is gratifying to note that the complete listing of maximal-rank regular subgroups of E_8 as given in **Table 1** is also obtained by an iterative application of Tables 14 and 15 in Ref [11].

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