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Construction of Strong Diophantine Quadruples and Pell Equation

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ABSTRACT: This paper aims at constructing strong Diophantine quadruples by employing the non zero distinct integer solutions of the Pell equations $y^2 = Dx^2 + 1$ and $y^2 = Dx^2 + 4$.

KEY WORDS: Strong Diophantine Quadruples, Pellian equation, integer solutions.

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I. INTRODUCTION

A set of positive integers $\{a_1, a_2 ......a_m\}$ is said to have the property D(n), $n \in Z - \{0\}$, if $a_i a_j + n$, a perfect square for all $1 \le i < j \le m$ and such a set is called a Diophantine m-tuples with property D(n).Many mathematicians considered the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n [1] and also for any linear polynomials in n. Further, various authors considered the connections of the problem of Diophantus, Davenport and Fibonacci numbers in [2-31].
In this communication, we aim at constructing strong Diophantine quadruples[32] with properties D(1) and D(4) by using the integer solutions of Pell equations $y^2 = Dx^2 + 1$ and $y^2 = Dx^2 + 4$.

II. METHOD OF ANALYSIS

2.1 Section-A

Consider the Pellian equation $y^2 = Dx^2 + 1$, D>0 and square free (1)
whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{y}_n = \frac{1}{2}[(\tilde{y}_0 + \sqrt{D}\tilde{x}_0)^{n+1} + (\tilde{y}_0 - \sqrt{D}\tilde{x}_0)^{n+1}]$$

$$\tilde{x}_n = \frac{1}{2\sqrt{D}}[(\tilde{y}_0 + \sqrt{D}\tilde{x}_0)^{n+1} - (\tilde{y}_0 - \sqrt{D}\tilde{x}_0)^{n+1}]$$

in which $(\tilde{x}_0, \tilde{y}_0)$ is the smallest positive integer solution of (1).

Let $a = \frac{\tilde{x}_n}{D^k}, b = D^{k+1}\tilde{x}_n$ be any two distinct numbers.

Observe that $ab + 1 = r^2$.
Thus (a, b) is a Diophantine double with property D(1).
Let c be any non-zero number such that

$$ac + 1 = \beta^2 \quad (2)$$

$$bc + 1 = \gamma^2 \quad (3)$$

Eliminating c between (2) and (3), we get

$b\beta^2 - a\gamma^2 = (b-a)$

The choice $\beta = X + aT, \gamma = X + bT$ (4)

leads to the Pell equation $X^2 = D\tilde{x}_n^2 T^2 + 1$

whose initial solution is

$T_0 = 1, X_0 = \tilde{y}_n$ (5)

Using (5) in (4) and employing either (2) or (3), we get

$c = \frac{\tilde{x}_n}{D^k}[1 + D^{2k+1}] + 2\tilde{y}_n$

Observe that (a,b,c) is a Strong Diophantine triple with the property D(1).

Using Euler's Solution {a,b,a+b+2r,4r(r+a)(r+b)} , we have

$\left\{ \frac{\tilde{x}_n}{D^k}, D^{k+1}\tilde{x}_n, \frac{\tilde{x}_n}{D^k}[1 + D^{2k+1}] + 2\tilde{y}_n, 4\tilde{y}_n[\tilde{y}_n^2 + D^{k+1}\tilde{x}_n\tilde{y}_n + \frac{\tilde{x}_n\tilde{y}_n}{D^k} + D\tilde{x}_n^2 \right\}$ as a strong diophantine quadruples with property D(1).

It is worth mentioning here that, when k=0, (6) represents Strong diophantine quadruples with property D(1) in integers, whereas when k>0, (6) represents Strong diophantine quadruples with property D(1) in rational numbes.

A few numerical illustrations are given below:

D	k	a	b	c	d
3	0	4	12	30	5852
13	0	180	2340	3818	6432579076
29	0	1820	52780	74202	$2.851125801\text{x}10^{13}$
31	0	273	8463	11776	$1.088290755\text{x}10^{11}$
3	4	15/81	3645	299472/81	804734736/81
17	2	8/289	39304	11377938/289	$4.956225978\text{x}10^{10}$

The above quadruples satisfy the relation $(a+b-c-d)^2 = 4(ab+1)(cd+1)$ .Thus the above quadruples are regular [33].

2.2 Section-B

Consider the Pellian equation $y^2 = Dx^2 + 4$, D>0 and square free (6)

whose general solution (x_n, y_n) is given by $(2\tilde{x}_n, 2\tilde{y}_n)$

Let $a = \frac{x_n}{D^k}, b = D^{k+1}x_n$ be any two distinct numbers.

Observe that $ab + 1 = \alpha^2$.

Thus (a, b) is a Diophantine double with property D(4).

Let c be any non-zero number such that

$ac + 1 = \beta^2$ (7)

$bc + 1 = \gamma^2$ (8)

Eliminating c between (7) and (8), we get

$b\beta^2 - a\gamma^2 = 4(b-a)$

The choice $\beta = X + aT, \gamma = X + bT$ (9)

leads to the Pell equation $X^2 = Dx_n^2 T^2 + 4$

whose initial solution is

$T_0 = 1, X_0 = 2\tilde{y}_n = y_n$ (10)

Using (10) in (9) and employing either (7) or (8), we get

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$$c=\frac{x_n}{D^k}[1+D^{2k+1}]+2y_n$$

Observe that (a,b,c) is a Strong Diophantine triple with the property D(4).
We have a well known result that the fourth tuple for the property D(4) is given by

$$d=a+b+c+\frac{1}{2}[abc+\alpha\beta\gamma]$$

$$d=\frac{x_n}{D^k}[2+D^{2k+1}]+D^{k+1}x_n+2y_n+Dx_n{}^2y_n+\frac{x_n{}^3}{2D^{k-1}}[1+D^{2k+1}]+\frac{1}{2}[y_n{}^2+\frac{x_ny_n}{D^k}][y_n+D^{k+1}x_n]$$

A few numerical illustrations are given below:

D	k	a	b	c	d
5	0	144	720	1508	156354184
19	0	78	1482	2240	258942640
24	0	2	48	70	6960
15	3	2/3375	101250	341772752/3375	$1.093759213\times10^{10}/3375$

III. REMARKABLE OBSERVATION

Let M_1, M_2 be two 3x3 matrices given by

$$M_1=\begin{bmatrix}0&0&-1\\1&0&2\\0&1&2\end{bmatrix} \text{ and } M_2=\begin{bmatrix}1&0&2\\0&0&-1\\0&1&2\end{bmatrix}$$

If (a,b,c) is any D(1) diophantine triple then $(a,b,c)M_1{}^n$ and $(a,b,c)M_2{}^n$ are also D(1) diophantine triples.

Illustration -1:

Consider the Diophantine triple $(a,b,c)=(x_n, 5x_n, 6x_n+2y_n)$ where $5x_n{}^2+1=y_n^2$ and

$$M_1=\begin{bmatrix}0&0&-1\\1&0&2\\0&1&2\end{bmatrix}$$

To start with $\begin{pmatrix}a & b & c\end{pmatrix}\begin{bmatrix}0&0&-1\\1&0&2\\0&1&2\end{bmatrix}=\begin{pmatrix}b & c & -a+2b+2c\end{pmatrix}$

Observe that $bc+1=(y_n+5x_n)^2$

$$b(-a+2b+2c)+1=(y_n+10x_n)^2$$

$$c(-a+2b+2c)+1=(3y_n+11x_n)^2$$

Therefore (b,c,-a+2b+2c) is a D(1) Diophantine triple.

Now, $\begin{pmatrix}a & b & c\end{pmatrix}\begin{bmatrix}0&0&-1\\1&0&2\\0&1&2\end{bmatrix}^2=\begin{pmatrix}c & -a+2b+2c & -2a+3b+6c\end{pmatrix}$

Observe that $c(-a+2b+2c)+1=(3y_n+11x_n)^2$

$$c(-2a+3b+6c)+1=(5y_n+17x_n)^2$$

$$d(-2a+3b+6c)+1=(7y_n+32x_n)^2$$

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Therefore (c,-a+2b+2c,-2a+3b+6c) is a D(1) Diophantine triple.

The repetition of the above process leads to the result that $(a,b,c)M_1{}^n$ is a D(1) Diophantine triple.

Illustration –2:

Consider the Diophantine triple $(a,b,c)=(x_n,3x_n,4x_n+2y_n)$ where $3x_n{}^2+1=y_n^2$ and

$$M_2=\begin{bmatrix}1 & 0 & 2\\ 0 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}$$

To start with $\begin{pmatrix}a & b & c\end{pmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}=\begin{pmatrix}a & c & 2a-b+2c\end{pmatrix}$

Observe that $ac+1=(y_n+x_n)^2$

$$a(2a-b+2c)+1=(y_n+2x_n)^2$$

$$c(2a-b+2c)+1=(3y_n+5x_n)^2$$

Therefore (a,c,2a-b+2c) is a D(1) Diophantine triple.

Now, $\begin{pmatrix}a & b & c\end{pmatrix}\begin{bmatrix}1 & 0 & 2\\ 0 & 0 & -1\\ 0 & 1 & 2\end{bmatrix}^2=\begin{pmatrix}a & 2a-b+2c & 6a-2b+3c\end{pmatrix}$

Observe that $a(2a-b+2c)+1=(y_n+2x_n)^2$

$$a(6a-2b+3c)+1=(y_n+3x_n)^2$$

$$(2a-b+2c)(6a-2b+3c)+1=(5y_n+9x_n)^2$$

Therefore (a,2a-b+2c,6a-2b+3c) is a D(1) Diophantine triple.

The repetition of the above process leads to the result that $(a,b,c)M_2{}^n$ is a D(1) Diophantine triple.

Thus, given any D(1) Diophantine triple, one may generate many Diophantine triple with property D(1) by employing the matrices M_1 and M_2 as illustrated above.

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