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Convergence of a Class of Non–Orthogonal Wavelet–Expansion in $L^p(R),\ 1 \le P < \infty$

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Abstract: It is known that a function in $L^2(R)$ can be represented by its wavelet expansion with convergence in $L^2(R)$–norm. But the case is different in $L'(R)$ norm. In this paper we shall discuss the convergence of non–orthogonal wavelet expansions in the $L^p(R)$ norm, $1 \le p \le \infty$. Moreover I have extended the results of Ahmed I Zayed [1] using the approximate identity. Our method and approach are different from those of Ahmed I. Zayed [1].

Keywords and Phrases: Approximate identity, Non orthogonal wavelet, Shannon's wavelet.

I. INTRODUCTION

S.Kelly, M.Kon and L. Raphael [1, 2] extended walter's results by proving pointwise convergence of orthogonal wavelet expansions in *n* dimensions. The key of their proofs is the following definition.

A. *Definition*

A bounded function $W:[0,\infty)\to R^+$ is a radial decreasing L^1-majorant of a given function *g* defined on *R* if $|g(x)| < w(|x|)$ and *w* satisfying the following conditions:

(i) $w \in L^1([0,\infty))$,

(ii) *w* is decreasing,

(iii) $w(0) < \infty$

The boundedness of *w* follows from (i) and (ii)

The summation kennel of the wavelet series $\sum_k < f, \phi^*_{m,k} > \phi_{m,k}(x)$ is given by

$$\sum_{k\to z} \phi_{m,k}(x)\overline{\phi_{m,k}}(y),\ \phi_{m,k}(x) = 2^{m/2}\phi(2^m x - k)$$

$$= 2^m \sum_{k\in z} \phi(2^m x - k)\phi\overline{(2^m y - k)}$$

$$= 2^m K_\phi(2^m x, 2^m y),$$

$$K_\phi(x,y) = \sum_{k\in z} \phi(x-k).\phi\overline{(y-k)}$$

Now we have

B. *Lemma*

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The summation kernel of the wavelet series is absolutely bounded by a radial decreasing L^1– majorant dilation kernel i.e.

$$\sum W(|x-k|)W(|y-k|) \le CW\left(\frac{|x-y|}{2}\right) \qquad x, y \in R$$

C is some constant depends on W.

Proof: – It is known that $|x-y| \le |x-k| + |y-k|$, either

$$|x-k| \ge \frac{1}{2} \text{ or } |y-k| \ge \frac{1}{2}|x-y|$$

So we have

$$\sum_{k\to z} W(|x-k|)W(|y-k|) \le \left(\frac{|y-k|}{2}\right)\left(\sum_{k\in z} W(|x-k|) + \sum_{k\in z} W(|y-k|)\right)$$

$$\le CW\left(\frac{|x-y|}{2}\right)$$

Hence the proof is completed.

Example(1) shows that there are some mother wavelets that do not satisfy above conditions.

1) *Example* : (The Discrete Shannon's Wavelet) The Shannon function ψ whose Fourier transform satisfies

$$\hat{\psi}(\xi) = \hat{x}_t(\xi)$$

where $t \in [-2\pi, -\pi] \cup [\pi, 2\pi]$ is called the Shannon wavelet. Thus this wavelet $\psi(t)$ can directly be obtained from the inverse Fourier transform of $\hat{\psi}(\xi)$ so that

$$\psi(t) = \frac{1}{2\pi}\int_R e^{+i\xi t}\hat{\psi}(\xi)\,d\xi$$

$$= \frac{1}{2\pi}\left[\int_{-2\pi}^{\pi} e^{-i\xi t}\,d\xi + \int_{\pi}^{2\pi} e^{i\xi t}\,d\xi\right]$$

$$= \frac{1}{\pi t}(\sin 2\pi t - \sin \pi t)$$

The associated scaling function is given by $\phi(t) = \frac{\sin \pi t}{\pi t}$. The summation kernel is seen to be

$$\frac{\sin \pi(t-y)}{\pi(t-y)} = \sum_{k=-\infty}^{\infty} \frac{\sin \pi(t-k)\sin \pi(y-k)}{\pi(t-k)\pi(y-k)}$$

It is clear that this kernel not belongs to $L^1(R)$. Hence this can not be absolutely bounded by radial decreasing L^1– Majorant function.

The point wise convergence of the Shannon wavelet series can be studied directly but it is very special case and of less interest. In this paper we shall study the pointwise convergence of wavelet expension in $L^P(R)$, $1 \le P < \infty$, associated with a class of mother wavelet that contains Shannon's wavelet as a special case. Although Ahmed I. Zayed [1] studied this problem but our methods and approach are different from those of [1].

Now, we have

C. *Definition*

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A multiresolution analysis of $L^2(R)$ is a sequence of closed subspaces $\{v_n\}_{n=-\infty}^{\infty}$ such that

(i) $\ldots\ldots \supset v_{-2} \supset v_{-1} \supset v_0 \supset v_1 \supset v_2 \ldots\ldots \supset v_n \supset v_{n+1} \ldots\ldots,$

(ii) $U_{n=-\infty}^{\infty} \; V_n$ is dense in $L^2(R)$, that is, $U_{n=-\infty}^{\infty} \; V_n = L^2(R)$ and $\bigcap_{n=-\infty}^{\infty} v_n = \{0\}$

(iii) $f(x) \in v_n$ if and only if $f(2x) = v_{n-1}$ for all $n \in z$,

(iv) V_0 is closed under integer translation, i.e., $f(x) = v_0$ implies that $f(x-k) \in v_0 \forall$ non negative integers k,

(v) there exists a function $\phi(x) = v_n$ such that $\{\phi_0, n = \phi(x-n) n \in z\}$ is an orthonormal basis for V_0 , that is

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |< f, \phi_{0,n} >|^2 \quad \text{for all } f \in v_0$$

The function ϕ is called the scaling function or father wavelet. If $\{v_n\}$ is a multiresolution analysis of $L^2(R)$ and v_0 is the closed subspaces generated by the integer translates of a single function ϕ, then we say that ϕ generates the multiresolution analysis.

We shall assume that the scaling function ϕ do not generate an orthonormal basis of v_0 or of W_0 (the orthogonal complement of v_0). This leads to the fact that condition (V) can be replaced by the weaker condition that $\{\phi(x-k)\}$ is a Riesz basis of v_0. Also we assume that the Fourier transform of ϕ has compact support. Since $\{\phi(x-k)\}$ is a Riesz basis of v_0 it means that $\{\phi_k^*(x)\}$ is a biorthonormal basis of $\{\phi(x-k)\}$ for $f \in v_0$ such that

$$f(x) = \sum_k < f, \; \phi_k^* > \phi_{0,k}(x)$$

similarly for $f \in v_m$, we have

$$f_m(x) = \sum_k < f, \phi_{m,k}^* > \phi_{m,k}(x)$$

Let us consider the class $S(R)$ of rapidly decreasing $c^\infty -$function on R such that

$$S(R) = \left\{ f : R \to R, \sup_{x \in R} \left(x^n \frac{d^m}{dx^m} f \right)(x) < \infty \right\} n, m \in N \cup \{0\}$$

Then for $f \in S(R)$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_R e^{-i\xi x} f(x) dx$$

It can be easily seen that if $f \in S(R)$ then $\hat{f}(\xi) \in S(R)$ is dense in $L^p(R), 1 \le P < \infty$. Also Fourier transform is isometric in $S(R)$. The inverse Fourier transform is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_R \hat{f}(\xi) e^{i\xi x} d\xi .$$

Let E denote the support of $\hat{\phi}$, is of the form.

$$E = U_{i=1}^n [a_i b_i] \cup O .$$

where O is a set of measure zero. Then for $f \in v_0$

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}}\int_E \hat{f}(\xi)e^{i\xi x}d\xi \\
&= \frac{1}{2\pi}\int_E\int_R f(t)e^{-it\xi}dt\, e^{+i\xi x}d\xi \\
&= \int_R f(t)\left\{\frac{1}{2\pi}\int_E e^{i\xi(x-t)}d\xi\right\}dt \\
&= \int_R f(t)K(t,x)dt
\end{aligned}
$$

where $$k(t,x) = \frac{1}{2\pi}\int_E e^{i\xi(x-t)}d\xi = k(x-t)$$

The integral in (1.3) is absolutely convergent by the Cauchy–Schwartz inequality because both f and K are in $L^2(R)$

D. *Definition*

If $\psi \in L^1(R)$ with $\hat{\psi}(0)=1$ and we define $\psi_n(x) = n\psi(nx)$ where $n\to\infty$. Then the sequences of functions $\{\psi_n\}_{n=1}^{\infty}$ is an approximate identity if:

(i) $\int_R \psi_n(x)dx = 1$ for all n.

(ii) $\sup_n \int_R |\psi_n(x)|dx < \infty$,

(iii) $\lim_{n\to\infty}\int_{|x|>\delta} |\psi_n(x)dx| = 0$ for every $\delta = 0$

1) *Remark* : If $0 \le \psi(x) \in S(R)$. then $\psi_n(x) = n\psi(nx)$ is an approximate identity.

By the hypothesis the reproducing kernel series $\sum_n \phi_n^*(t)\phi(x-n)$ converges absolutely and uniformly for all x and t to a function $q_0(t,x)$. Thus, we have

$k(t,x) = \sum_n \phi_n^*(t)\phi(x-n) = q_0(t,x)$ almost everywhere. Similarly we can write

$$
\begin{aligned}
q_m(t,x) &= \sum_n \phi_{m,n}^*(t)\phi_{m,n}(x) \\
&= 2^m \sum_n \phi_n^*(2^m t)\phi(2^m x - n) \\
&= 2^m q_0(2^m t, 2^m x) \\
&= \lambda q_0(\lambda t, \lambda x), \quad \lambda = 2^m
\end{aligned}
$$

Now we have

E. *Lemma*

If $q_0(t,x) \in L^1(R)$ with $\int_R q_0(t,x)dt = 1$. Then $q_m(t,x)$ is an approximate identity if.

1. $\int_R q_m(t,x)dt = 1$ for all m,
2. $\sup_m \int_R |q_m(t,x)|dt < \infty$

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3. $\lim_{m\to\infty}\int_{|x-t|>\delta}|q_m(t,x)|\,dt=0$ for every $\delta>0$.

1) *Proof* **:** We observe that

$$\int_R q_m(t,x)\,dt=\int_R \lambda q_0\{\lambda(x-t)\}\,dt=\int_R q_0(\lambda(|x-t|)\,d(\lambda|x-t|))=1$$

2. We have

$$q_0(\xi)=\frac{1}{2\pi|\xi|}\geq\frac{1}{\sqrt{2\pi}}\left|\frac{e^{-i\xi b}-e^{-i\xi a}}{-i\xi}\right|=\left|\frac{1}{\sqrt{2\pi}}\int_a^b e^{-i\xi x}dx\right|,$$

$$\lambda_{q_0}(\lambda|\xi|)\geq\sum_{i=1}^m\gamma_i\left|\frac{e^{-i\xi b}-e^{-i\xi a}}{\sqrt{2\pi}(-i\xi)}\right|=\sum_{i=1}^m\gamma_i\int_{\lambda a_i}^{\lambda b_i}x_{\Delta i}(x)$$

If $f\in L^1(R)$ then $\lim_{m\to\infty}\sum_{i=1}^m\gamma_i\chi_{\Delta i}(x)$ is in $L^1(R)$. Given $\in>0$, find $\sum_{i=1}^m\gamma_i\chi_{\Delta i}(x)$ such that

$$\int_R\left|f(x)-\sum_{i=1}^m\gamma_i\chi_{\Delta i}(x)\right|dx<\varepsilon/2$$

So we have

$$\left|\hat{f}(\xi)\right|\leq\left|\frac{1}{\sqrt{2\pi}}\int_R e^{-i\xi x}f(x)\,dx\right|$$

$$\leq\frac{1}{\sqrt{2\pi}}\int_R\left\{\left|f(x)-\sum_{i=1}^m\gamma_i\chi_{\Delta i}(x)\right|dx+\left|\sum_{i=1}^m\gamma_i\chi_{\Delta i}(x)\right|\right\}e^{-i\xi x}dx$$

$$\leq\frac{\varepsilon}{2\sqrt{2\pi}}\frac{\sum\gamma_i}{|\xi|}<\frac{\varepsilon}{2\sqrt{2\pi}}$$

Hence $\lambda q(\lambda|\xi|)$ is bounded for all ξ and λ. This proves 2.

3. We have

$$\int_{|x-t|>\delta}q_m(t,x)\,dt=\int_{|x-t|>\delta}\lambda q_0(\lambda|x-t|)\,dt$$

$$=\int_\delta^\infty\lambda q_0(\lambda|x-t|)\,dt+\int_{-\infty}^{-\delta}\lambda q_0(\lambda|x-t|)\,dt$$

substituting $x-y=\lambda(x-t)$

$$\lim_{m\to\infty}\int_{m\delta}^\infty q_0(x-y)\,dy+\int_{-\infty}^{-m\delta}q_0(x-y)\,dy=0$$

2) *Remark* : Let $0\leq q_0(t,x)\in S(R)$. Then $q_m(t,x)=\lambda q_0(\lambda|x-t|)$ is an approximate identity.

F. ***Lemma***

If $f\in L'(R)$ and $q_m(t,x)\in S(R)$ then $q_m(t,x)*f\in S(R)$

Proof**:** We have

$$(q_m(t,x)*f)=\int_R q_m(x-t-y)f(y)\,dy$$

$$=\int_R q_m(y-t)f(x-y)\,dy$$

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or $$\frac{d^n}{dx^n}\left(q_m(t,x)*f\right)=\int_R q_m(y-t)\frac{d^n}{dx^n}f(x-y)\,dy$$

or $$|x|^n\frac{d^n}{dx^n}\left(q_m(t,x)*f\right)=|x|^n\int_R f(x-y)\frac{d^n}{dx^n}q_m(y-t)\,dy,$$

substituting $x-y=z$, it gives that

$$=\int_R f(y)|x|^n\frac{d^n}{dx^n}q_m(x-y-t)\,dy,$$

since $|x-y-t|\le|x-t|+|y|\le\frac{3|x-t|}{2}$, so from above we obtain

$$=\int_{|y|>\frac{|x-t|}{2}} f(y)|x|^n\frac{d^n}{dx^n}q_m(x-y-t)\,dy,$$

$$+\int_{|y|\le\frac{|x-t|}{2}} f(y)|x|^n\frac{d^n}{dx^n}q_m(x-y-t)\,dy\to 0$$

Hence the proof is completed.

II. MAIN RESULTS

A. *Theorem*

If $\{q_m(t,x)\}_{m=1}^{\infty}$ is an approximate identity then $\lim_{m\to\infty}\|f*q_m(t,x)-f\|_p=0$ for every $f\in L^P(R),\ 1\le P<\infty$

Proof**:** Let us consider

$$\left[\int_R\left|\left(q_m(t,x)*f\right)(x)-f(x)\right|^P dx\right]^{1/P}$$

$$=\left[\int_R dx\left|\int_R q_m(x-t-y)f(y)\,dy-f(x)\right|^P\right]^{1/P}$$

$$=\left[\int_R dx\left|\int_R q_m(y-t)f(x-y)\,dy-f(x)\right|^P\right]^{1/P}$$

Since $f(x)=\int_R f(x)q_m(y-t)\,dy$, so we get

$$\left[\int_R dx\left|\int_R q_m(y-t)\{f(x-y)-f(x)\}\,dy\right|^P\right]^{1/P}$$

$$\le\left[\int_R dx\int_{|y-t|>\delta}|q_m(y-t)|^P|f(x-y)-f(x)|^P\,dy\right]^{1/P}$$

$$+\left[\int_R dx\int_{|y-t|\le\delta}|q_m(y-t)|^P|f(x-y)-f(x)|^P\,dy\right]^{1/P}$$

$$\le\int_{|y-t|>\delta}dy\,|q_m(y-t)|\left[\int_R dx\,|f(x-y)-f(x)|^P\right]^{1/P}$$

$$+\int_{|y-t|\le\delta}dy\,|q_m(y-t)|\left[\int_R dx\,|f(x-y)-f(x)|^P\right]^{1/P}$$

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$$\leq \int_{|y-t|>\delta} dy \left|q_m(y-t)\right| \left(2\|f\|_P\right)$$

$$+\int_{|y-t|\leq\delta} dy \left|q_m(y-t)\right| \sup_{|y|<\delta}\left[\int_R \left|f(x-y)-f(x)\right|^P dx\right]^{1/P} \to 0$$

as $m \to \infty$

Hence the proof is completed.

B. *Theorem*

If $q_m(t,x)$ is an approximate identity and $f \in L^P(R)$, then the wavelet series.-

$$f_m(x) = \sum_k < f, \phi^*_{m,k} > \phi_{m,k}(x)$$

Converges to $f(x)$ as $m \to \infty$ at every point of continuity of $f(x)$

1) *Proof* **: –** The projection of $f \in L^2(R)$, on the space V_m is given by

$$f_m(x) = \int_R f(t) q_m(t,x) dt$$

$$= \int_R f(t) q_m(x-t) dt$$

$$= (f * q_m)(x) \to f(x), \quad (q_m \text{ is an approximate identity})$$

Also we have

$$f_m(x) = \int_R f(t) q_m(x-t) dt$$

$$= \sum_n \int_R f(t) \phi^*{}_{m,n}(t) \phi_{m,n}(x) dt$$

$$= \sum_n < f, \phi^*{}_{m,n} > \phi_{m,n}(x)$$

Thus if $f \in L^P(R)$, $q_m(x) \in S(R)$ then $q_m{}^* f \in S(R)$ and $S(R)$ is dense in $L^P(R), 1 \leq p < \infty$ then

$$\|f_m(x) - f(x)\|_{L^P(R)} \to 0 \text{ as } m \to \infty$$

This proves the theorem.

C.*Theorem*

If $f \in L^1(R) q_m(t,x) \in L^P(R)$, then

$$\|f * q_m(t,x)\|_p \leq \|f\|_q \|q_m(t,x)\|_P \quad 1 \leq P < \infty$$

Proof**:** If $f \in L'(R), du = |f(x)| dx$ is a finite Borel measure, so we have

$$\mu_N = \sum_{i=1}^{N} c_i^N \delta_{\beta_i(N)} \to \mu \text{ weakly}$$

Now Consider for $1 \leq p < \infty$,

$$\|q_m(t,x) * f\|_p = \left[\int_R \left|q_m(x-t-y) f(y)\right|^P dy\right]^{1/P}$$

$$\leq \left[\int_R \left|q_m(x-t-y) d\mu_n(y)\right|^P dy\right]^{1/P},$$

$$\left[\int_R d\mu_n f(x) \to \int_R f(x)\,d\mu\right]$$
$$\leq \left\| \sum_{i=1}^{N} q_m\left(x-t-\beta_i^n\right) C_i^N \right\|_P$$
$$\leq \sum_{i=1}^{N} C_i^N \left\| q_m(t,x) \right\|_P$$
$$= \left\| q_m(t,x) \right\|_P \int_R d\mu_n(y)$$
$$\leq \left\| q_m(t,x) \right\|_P \left\| f \right\|,$$

1) *Remark*: For $P = \infty$, we get

$$\left| q_m(t,x) * f \right| \leq \| f \|_\infty \left\| q_m(t,x) \right\|_1$$

In view of theorem E, we get the following corollary.

D. Corollary

If $f \in L'(R)$ and $q_m(t,x) \in L^P(R)$ then

$$\| f_m(x) \|_P \leq \| f \|_1 \left\| q_m(t,x) \right\|_P, \qquad 1 \leq P < \infty$$

1) *Remark* : If $f \in L'(R)$, $q_0(x) \in S(R)$ then

$$\left(f * q_0\right)(x) = \int_R f(t) q_0(x-t)\,dt$$
$$= \sum_n \int_R f(t) \phi_n^*(t) \phi(x-n)\,dt = \sum_n < f \phi_n^* > \phi(x-n)$$
$$= f(x)$$

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