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Entanglement Increase in Higher Dimensions

Mohammad Alimoradi Chamgordani[1*], Henk Koppelaar[2]

Department of Physics, Shahid Chamran University, Ahvaz, Iran[1]

Department of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, Delft, The Netherlands [2]

ABSTRACT: Entanglement of generalized Quasi-Bell states is studied and numerically compared with I-concurrence (Ic), D-concurrence (Dc), negativity (N) and mutual information ($S^{A:B}$) measures. We illustrate denoted measures reveal more entanglement as the generalized two-qutrit (qudit) Quasi-Bell state tend to be more complete in $d\times d$ Hilbert spaces, $d\in\{3\ ,\ 4\}$. Entanglement and correlation of two qutrit- and two qudit-Werner states are determined analytically by employing negativity and mutual information and then compared numerically.

KEYWORDS: Quantum information,Entanglement, I-concurrence, D-concurrence,Negativity, Quasi-Bell states, Werner states

I. INTRODUCTION

Entanglement is an outstanding feature of quantum theory which distinguishes it from classical physics [1]. Bell's inequalities explain these distinctions more quantitative [2,3].Entanglement measures of pure states will be zero if the pure ket-state is separable. But for maximally entangled states (Bell states) these measures exhibit maximum values. Entanglement properties of some special quantum states like superposition of coherent states [4-6] and two-qubit Werner states have been investigated [7].Recent studies have demonstrated that quantum systems in higher dimensions may improve the efficiency of quantum information protocols, security of quantum cryptography, and quantum channel capacities [8-12].Because higher dimensional entangled states allow the realization of new types with higher capacity of quantum communication protocols [13].

In 2008, a geometric approach wasapplied to detect entanglement in multi-qubit systems [14].Later the same method was used to detect entanglement for paired systems more complicated than qubits [15].Before, the requirementsof use of qutrits for higher dimensional entangledsystems [16-19]as well as its experimental verification has been investigated. Recently entanglement in multipartite systems in high dimension was studied [20,21]. We generally show in this paper that bipartite systems in higher dimensional Hilbert spaces show higher entanglement. Entanglement of Werner states in higher dimensions has not been investigated yet.

The main purpose of this work is to quantify and compare entanglement of two-qutrit and two-quditgeneralized Werner states as well as generalized Quasi-Bell states by applying various entanglement measures.We investigate entanglement features of a family of Werner states because of their beneficial usage [22,23]for two-qutrit Werner states, [24] and two-qudit Werner states.

The organization of this paper is as follows. Some popular measures of entanglement are introduced in section II. In section III entanglement of generalized bipartite Quasi-Bell states is considered. In section IV entanglement of two-qutrit and two-qudit Werner states are investigated and studied. Finally, section V concludes the work.

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II. ENTANGLEMENT MEASURES

I-concurrence (Ic) is an appropriate measure of entanglement in case of systems of higher dimension [25]. For a pure bipartite state $|\psi^{AB}\rangle$ the Ic is defined by:

$$\mathrm{Ic}\left(\left|\psi^{AB}\right\rangle\right)=\sqrt{2\left(1-tr\left(Tr_B\left|\psi^{AB}\right\rangle\left\langle\psi^{AB}\right|\right)^2\right)}, \tag{1}$$

in which tr and Tr_B respectively denote trace and partial trace with respect to part B. The minimum for Ic measures is equal to zero while for $d\times d$-dimensional systems the maximum value of Ic is $\sqrt{2(d-1)/d}$.

D-concurrence (Dc) [26] and von Neumann entropy (S) [27] used for measuring entanglement of $|\psi^{AB}\rangle$ in higher dimensions, are respectively given by:

$$\mathrm{Dc}\left(\left|\psi^{AB}\right\rangle\right)=\sqrt{Det\left(I^A-Tr_B\left|\psi^{AB}\right\rangle\left\langle\psi^{AB}\right|\right)}, \tag{2}$$

$$\mathrm{S}=-tr\left(\rho^A\ln\rho^A\right), \tag{3}$$

in which ρ^A (ρ^B) is derived from density matrix of $|\psi^{AB}\rangle$ by tracing out the other part B (A). $Det(I^A-\rho^A)$ is the determinant of the matrix representation of the $I^A-\rho^A$ operator. In quantum systems as well in classical systems, information is embedded in the correlation between the subsystems. Therefore to present both a description for entanglement and a classical correlation of a bipartite system, Mutual Information [27,28]is defined as follows:

$$\mathrm{S}^{A:B}=\mathrm{S}^A+\mathrm{S}^B-\mathrm{S}^{AB}, \tag{4}$$

Where S^A and S^B are the von Neumann entropies of subsystems A and B and S^{AB} is the joint von Neumann entropy of the composite quantum system AB. Negativity for a bipartite density operator ρ^{AB} is given[28] by:

$$\mathrm{N}=\frac{\left\|\rho^{T_A}\right\|-1}{d-1}, \tag{5}$$

Where $\left\|\rho^{T_A}\right\|$ is the trace norm ofthe partial transpose of ρ^{AB} with respect to subsystem A and d is equal to $\min\{\dim(A),\dim(B)\}$.

III. ENTANGLEMENT OF BIPARTITE GENERALIZED QUASI-BELL STATES

We focus on finite-dimensional bipartite quantum systems, i.e., systems composed of two distinct subsystems, described by the Hilbert space H = $H_1 \otimes H_2$. Quasi-Bell states have shown wide applicability in different fields of quantum information and computation [29-32]. In the sequel we investigate the entanglement property of two- qutrit and qudit generalized Quasi-Bell states in that order.

3.1. Entanglement of Two-Qudit Generalized Quasi-Bell States

Entangled pure two-qudit generalized Quasi-Bell states in a 4×4-Hilbert space are as follows:

$$|\psi\rangle=\frac{1}{\sqrt{N_\psi}}\sum_{i=0}^{3}|c_{ii}||i,i\rangle,\ N_\psi=\sum_{i=0}^{3}|c_{ii}|^2, \tag{6}$$

$$|\psi'\rangle=\frac{1}{\sqrt{N_{\psi'}}}\sum_{i=1}^{3}|c_{ii}||i,i\rangle,\ N_{\psi'}=\sum_{i=1}^{3}|c_{ii}|^2, \tag{7}$$

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$$|\psi''\rangle = \frac{1}{\sqrt{N_{\psi''}}}\sum_{i=2}^{3}|c_{ii}||i,i\rangle,\ N_{\psi''} = \sum_{i=2}^{3}|c_{ii}|^2, \tag{8}$$

Maximum values of the measures Dc, N, S, Ic and $S^{A:B}$ for pure states $|\psi\rangle$, $|\psi'\rangle$ and $|\psi''\rangle$ are illustrated in Fig. 1. According to Fig. 1 all the employed entanglement measures reveal stronger entanglement if the two-qudit Quasi-Bell states tend to be complete (changing $|\psi''\rangle$ to $|\psi\rangle$) in a 4×4 Hilbert space. Although the reviewed measures show different numerical values for mentioned two-qudit states they also exhibit a common behavior of increasing entanglement.

3.2. Entanglement of Two-Qutrit Generalized Quasi-Bell States

Normalized two-qutrit generalized Quasi-bell states in a 3×3-Hilbert space are considered as follows:

$$|\varphi_1\rangle = \frac{1}{\sqrt{N_{\varphi_1}}}\sum_{j=0}^{2}|d_{jj}||j,j\rangle,\ \ N_{\varphi_1} = \sum_{j=0}^{2}|d_{jj}|^2, \tag{9}$$

$$|\varphi_2\rangle = \frac{1}{\sqrt{N_{\varphi_2}}}\sum_{j=1}^{2}|d_{jj}||j,j\rangle,\ N_{\varphi_2} = \sum_{j=1}^{2}|d_{jj}|^2, \tag{10}$$

Values of denoted entanglement measures for generalized two-qutrit Quasi-Bell states are shown in Fig. 2 to become maximal if the two-qutrit Quasi-Bell states tend to be completed ($|\varphi_2\rangle$ changes to $|\varphi_1\rangle$) in a 3×3 Hilbert space. With regard to Figs. 1 and 2, the difference in the maximum values of measures corresponds to the various types of generalized Quai-Bell states defined in 3×3 and 4×4 Hilbert spaces. For states $|\psi\rangle$, $|\psi'\rangle$, $|\psi''\rangle$ in a 4×4 Hilbert space and states $|\varphi_2\rangle$, $|\varphi_1\rangle$ in a 3×3 Hilbert space, negativity measures' maxima are 0.33, 0.66, 1 and 0.5, 1 respectively. Concluding, negativity is an appropriate measure to quantify entanglement of bipartite states in a Hilbert space because it maps different values of entanglement to different states. But it has not got eligible success to compare entanglement of denoted bipartite pure states in different Hilbert spaces. For example $\mathrm{N}(|\varphi_1\rangle) > \mathrm{N}(|\varphi_2\rangle)$ and $\mathrm{N}(|\psi\rangle) > \mathrm{N}(|\psi'\rangle) > \mathrm{N}(|\psi''\rangle)$ but $\mathrm{N}(|\varphi_1\rangle) > \mathrm{N}(|\psi'\rangle)$ and $\mathrm{N}(|\varphi_2\rangle) > \mathrm{N}(|\psi''\rangle)$ are not reasonable and are in contrast with Ic results. This argument is also true for S and $S^{A:B}$. The Dc measure does not reveal eligible difference in values of entanglement related to states $|\psi\rangle$, $|\psi'\rangle$, $|\psi''\rangle$ and $|\varphi_2\rangle$, $|\varphi_1\rangle$ in 4×4 and 3×3 Hilbert spaces respectively, and also does not distinguish entanglement of denoted states well. By inspection of Figs. 1 and 2, it is appropriate to use the Ic measure to compare entanglement of bipartite pure states in higher dimension. Because this measure maps different degrees of entanglement to various bipartite pure states in a Hilbert space and also it compares entanglement of different bipartite pure states in different Hilbert spaces by different values attribution.

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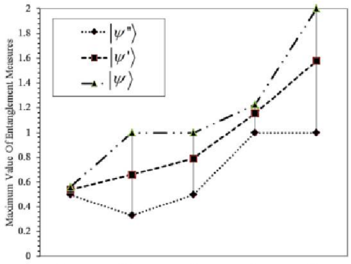

Fig. 1. Maximum degrees of entanglement measures for generalized two qudit Quasi-Bell states $|\psi\rangle$, $|\psi'\rangle$ and $|\psi''\rangle$.

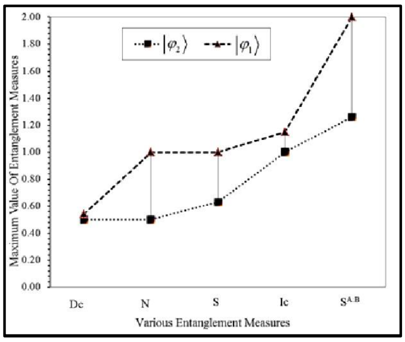

Fig. 2. Maximum degrees of entanglement measures (N, Ic, Dc, S, $S^{A:B}$) for generalized two-qutrit Quasi-Bell states $|\varphi_1\rangle$ and $|\varphi_2\rangle$.

IV. ENTANGLEMENT OF WERNER STATES IN 3x3 AND 4x4 HILBERT SPACES

We investigate entanglement properties of two types of Werner states in higher dimensions. Correlation and entanglement are quantified analytically with mutual information and negativity for two qutrit- and two qudit-Werner states. Then the results will be compared numerically.

4.1. Two-qutrit Werner States

A two-qutrit Werner state is defined as follows:

$$\rho_{wer}^{\chi} = p|\chi\rangle\langle\chi| + \frac{1-p}{9} I_{9\times 9}\,, \tag{11}$$

In which $|\chi\rangle$ is normalized and is defined as follows:

$$|\chi\rangle = \cos\theta|02\rangle + e^{-i\varphi}\sin\theta|20\rangle\,. \tag{12}$$

The measures N and $S^{A:B}$ for (8) that depend on p and θ states, are computed as follows:

$$N = \frac{1}{18}\Big[\left|9p\cos^2\theta - p + 1\right| + 5\left|1-p\right| + \left|9p\sin^2\theta - p + 1\right| + \left|\frac{9}{2}p\left|\sin 2\theta\right| + p - 1\right| + \left|\frac{9}{2}p\left|\sin 2\theta\right| - p + 1\right| - 9\Big], \tag{13}$$

$$S^{A:B} = C'Log_3 C'A'B' + p\sin^2\theta Log_3\frac{B'}{A'} + pLog_3 A' + \frac{8C'}{3}Log_3\frac{C'}{3} + \frac{1+8p}{9}Log_3\frac{1+8p}{9} \tag{14}$$

The above functions are independent of φ and A', B' and C'

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$$A' = \frac{1-p+3p\cos^2\theta}{3},\ C' = \frac{1-p}{3},\ B' = \frac{1-p+3p\sin^2\theta}{3}. \quad (15)$$

For ρ_{wer}^{χ} the behavior of N and $S^{A:B}$ is non-monotonic and with poorly regular oscillation. The lack of monotonicity and irregularity of oscillations correspond to the combination of Quasi-Bell states $|\varphi\rangle$ and the completely separable mixed states $I_{9\times 9}$ with probability coefficient *p*. This inconsonance of entanglement is observable in Figs. 3 and 4. According to Figs. 3 and 4, the results of entanglement for N and $S^{A:B}$ measures are similar. Maximum value of N and $S^{A:B}$ in these figures is in accordance with the maximum value of N and $S^{A:B}$ for states $|\varphi_2\rangle$ in Fig. 2. Indeed for p=1 the state (8) is a representation of two-qutrit Quasi-Bell state $|\varphi_2\rangle$. For the other values of *p*, due to the increase of ρ_{wer}^{χ} impurity, maximum level of entanglement will decrease as shown in Fig. 4.

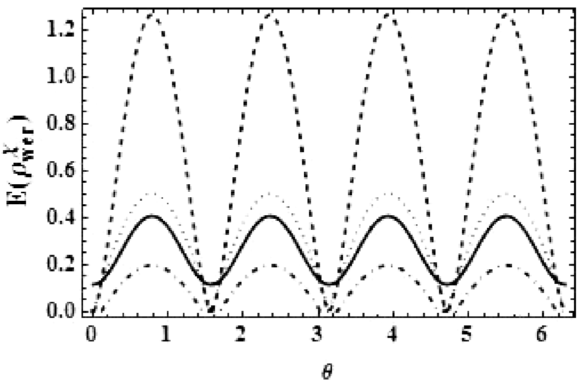

Fig. 3. Entanglement of ρ_{wer}^{χ} as a function of θ; $S^{A:B}$ (solid line) for p=0.5 and (dashed line) for p=1; N (dashed-dotted line) for p=0.5 and (dotted line) for p=1.

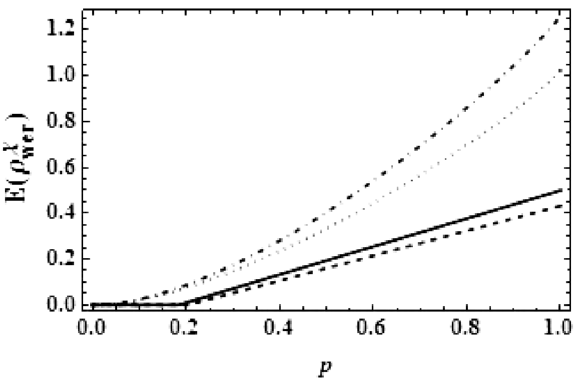

Fig. 4. Entanglement of ρ_{wer}^{χ} as function of *p*; N (solid-line) and $S^{A:B}$ (dashed-dotted line) for $\theta = \frac{(2n+1)\pi}{4}$ (*n*=0, 1, 2,…), N (dashed-line) and $S^{A:B}$ (dotted-line) for $\theta = \frac{\left[6(m-1)+a_f\right]\pi}{3}$ (m = 1, 2, 3,...and a_f=1, 2, 4, 5).

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4.2. Two-qudit Werner States

A generalized two-qudit Werner state is defined as follows:

$$\rho_{wer}^{\chi'} = p|\chi'\rangle\langle\chi'| + \frac{1-p}{16} I_{16\times16} \qquad (16)$$

in which $I_{16\times16}$ is fully separable mixed state and non-normalized $|\chi'\rangle$ is defined as:

$$|\chi'\rangle = |\alpha||00\rangle + |\beta||11\rangle + |\gamma||22\rangle + |\delta||33\rangle \qquad (17)$$

The behavior of N and $S^{A:B}$ measures and their extremes depend on the p parameter in (13). Taking into view Figs. 5 and 6, for p = 1 the entanglement for state (13) close to the coefficients α, β, γ and δ is exactly similar to entanglement of states 6-8. Their maximum values for N and $S^{A:B}$ are showed in Fig. 1. Indeed the maximum points of N and $S^{A:B}$ in Figs. 5 and 6 are in accordance with the maximum values of these measures in the case of $|\psi\rangle$, $|\psi'\rangle$ and $|\psi''\rangle$ states in Fig. 1. For the other values of the p entanglement starts to reach its minimum value based on the behavior of N and $S^{A:B}$ illustrated in Figs. 5 and 6. Comparison with entanglement measures N and $S^{A:B}$ shows their behavior in describing entanglement of state (13) is very similar.

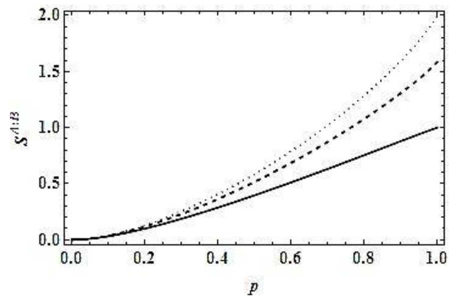

Fig. 5. $S^{A:B}$ of ρ_{wer}^{χ} as function of p; for α=δ=β=γ=1 (dotted line), for α=β=γ= 1, δ =0 (dashed line), for β=γ=1, α=δ= 0 (solid line).

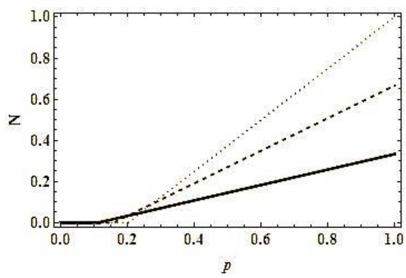

Fig. 6. N of ρ_{wer}^{χ} as function of p; for α=δ=β=γ=1 (dotted line), for α=β=γ= 1, δ =0 (dashed line), for β=γ=1,α=δ= 0 (solid line).

V. DISCUSSION AND CONCLUSION

We have shown that more than a single entanglement measure could be employed to quantify entanglement of a bipartite system. By employing Ic, Dc, N and $S^{A:B}$ measures, we have illustrated that if the bipartite generalized Quasi-Bell stats tend to be more complete in $d\times d$ Hilbert spaces $d\in\{3, 4\}$, they will reveal more entanglement. It has been shown that entanglement of bipartite two qudit Quasi-Bell states in a 4×4 Hilbert space is larger than two qutrit Quasi-Bell states in a 3×3 Hilbert space. The best measure which confirms this claim is I-concurrence. As an example of two qutrit and two qudit states, this was demonstrated for two qutrit and two qudit Werner states with negativity and mutual information as well. For denoted Werner states, we obtained negativity and mutual information analytically and

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compared the results numerically against the *p* parameter. They revealed similar treatment of quantifying entanglement. Using negativity and mutual information, we demonstrated that by changing the *p* parameter, entanglement of Werner states will change betweenextremum values of N and $S^{A:B}$ of denoted Quasi-bell states.

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