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Numerical Solution of Fractional Variational Problems Using Direct Haar Wavelet Method

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ABSTRACT: This paper presents a clear procedure for the fractional variational solution via Haar wavelet technique. The fractional derivative is defined in the Riemann-Liouville sense. The fractional variational problem is solved by means of the direct method using the Haar wavelet and the problem will be reduced to the solution of an algebraic equations. The numerical solution for the class of problem considered can be obtained directly from the functional and there is no need to solve the fractional Euler-Lagrange equation. The examples are included in order to demonstrate the validity and applicability of the suggested approach.

KEYWORDS: Haar wavelet method, Fractional calculus, Calculus of variation, Fractional calculus of variation.

I. INTRODUCTION

The use of fractional calculus of modeling physical system has been widely considered in the last decades, [1]. Although, the concept of the fractional derivatives was introduced already in the middle of the 19^{th} century by Riemann and Liouville, [2]. The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [3] published in 1974. After that, the number of publications about the fractional calculus has rapidly increased. The reason for this is that the same physical processes as a anomalous diffusion, complex viscoelastisity, behaviour of mehatronic and biological systems, rheology, etc. can be described adequately by classical models, [2].

A fractional calculus of variations problem is a subtopic of fractional calculus and it is a problem in which either the objective functional or the constraint equation or both contain at least one fractional derivative term, [4]. This occurs naturally in many problems of physics, mechanics and engineering in order to provide more accurate models of physical phenomena (see [5-13]), However, the fractional calculus of variations is a new field, ;Its starting point appear to be the references [14], [15] where Riewe developed the nonconcentrative Lagrangian, Hamiltonian, and other concepts of classical mechanics using fractional derivative, [16]. Particularly, a fractional calculus of variation concerns the variational principles on functionals involving fractional derivative as we mention above and this leads to the statement of fractional Euler-Lagrange equations (see [14], [4], [17]). Fractional Euler-Lagrange equations are difficult to solve explicitly and consequently it is of interest to develop efficient numerical schemes for such dynamical systems. In this paper, we shall use the direct Haar wavelet method for a class of fractional variational problems. Haar wavelet theory has been innovated and applied to various fields in engineering ([18]-[25]), and have proved to be a wonderful mathematical tool.

The idea of this paper is to introduce Haar wavelets, then present a direct method for solving fractional problems via Haar wavelets. The procedure begins by assuming the admissible functions by Haar wavelets with coefficients to be determined, then establishing an operational matrix for performing integration and finding the necessary condition for exterimization, solving the resulting algebraic equation yields the Haar coefficients. This indicates that for the class of problems that will be considered, the numerical solution can be obtained directly from the functional, and there is no need to solve the fractional Euler-Lagrange equations.

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II. Fractional Derivative and Integration

In this section, we shall review the basic definitions and properties of fractional integral and derivative, which are used further in this paper, [1].

Definition (1):

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, is defined as:

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1} f(x)\, dx, \ \alpha > 0,\ x > 0$$

$$I^0 f(t) = f(t).$$

Definition (2):

The Riemann-Liouville fractional derivative operator of order $\alpha > 0$, is defined as:

$$_0D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-x)^{n-\alpha-1} f(x)\, dx$$

where n is an integer and $n - 1 < \alpha \le n$.

Definition (3):

The Caputo fractional derivative operator of order α, is defined as:

$$^cD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1} \frac{d^n}{dx^n} f(x)\, dx$$

where n is an integer and $n - 1 < \alpha \le n$.

Caputo fractional derivative has an useful property:

$$I^{\alpha}\, {}^cD_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}$$

where n is an integer and $n - 1 < \alpha \le n$.

III. Haar Wavelet

The orthogonal set of Haar wavelets $h_i(t)$ is a group of square waves with magnitude of ±1 in certain intervals and zeros elsewhere. The orthogonal basis $\{h_n(t)\}$ of Haar wavelets for the Hilbert space $L_2[0,1]$ consists of:

$h_n(t) = h_1(2^j t - k/2^j)$, $n = 2^j + k$, $j \ge 0$, $0 \le k \le 2^j$, $k \in$

where:

$h_0(t) = 1,\ 0 \le t < 1$

$$h_1(t) = \begin{cases} 1, & 0 \le t < 0.5 \\ -1, & 0.5 \le t < 1 \end{cases}$$

each Haar wavelet h_n has the support $(2^j k, 2^{-j}(k+1))$, so that it is zero elsewhere in the interval [0,1). Interestingly, as n inverses, the Haar wavelets become more and more localized. Therefore, $\{h_{n+1}(t)\}$ form m-local basis. Any function $f(t) \in L_2[0,1)$ can be expanded in Haar series (see [20]), as:

$$f(t) = \sum_{i=0}^{\infty} c_i h_i(t),\ n = 2^j + k,\ j \ge 0,\ 0 \le k \le 2 \tag{1}$$

where the Haar coefficients c_i, i = 0, 1, …; are given by:

$$c_i = 2^j \int_0^1 f(t) h_i(t)\, dt$$

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which are determined such that the following integral square error e is minimized:

$$e = \int_0^1 \left[f(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt \ , \ m = 2^j, j \in \{0\} \cup \mathbb{N}$$

by applying the orthogonal property of Haar wavelet:

$$\int_0^1 h_\ell(t) h_i(t)\, dt = \begin{cases} 2^{-j}, & i = \ell \\ 0, & i \neq \ell \end{cases}$$

The series in eq.(1) contains an infinite number of terms. If $f(t)$ is piecewise constant or may be approximated as a piecewise constant, then the sum in eq.(1) may be determined after m-terms:

$$f(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) \ = C_m^T \mathrm{H_m(t)} = \hat{f}(t) \tag{2}$$

where $m = 2^j$, the superscript T indicates the transposition, $\hat{f}(t)$ denotes the truncated sum. The Haar coefficient vector C_m and the Haar function vector $H_m(t)$ are defined as:

$$C_m \triangleq [c_0, c_1, \ldots, c_{m-1}]^T$$

$$H_m(t) \triangleq [h_0(t), h_1(t), \ldots, h_{m-1}(t)]^T \tag{3}$$

Taking the collection points as follows:

$$t_i = \frac{(2i-1)}{2m}, \ i = 1, 2, \ldots, m$$

we define the m-sequence Haar matrix $\Phi_{m\times m}$ as:

$$\Phi_{m\times m} \triangleq \left[H_m\left(\frac{1}{2m}\right) H_m\left(\frac{3}{2m}\right) \ldots H_m\left(\frac{(2m-1)}{2m}\right) \right]$$

Correspondingly, we have:

$$\hat{f}_m = \left[\hat{f}\left(\frac{1}{2m}\right) \hat{f}\left(\frac{3}{2m}\right) \ldots \hat{f}\left(\frac{(2m-1)}{2m}\right) \right] = C_m^T \Phi_{m\times m}$$

Because of the m-square matrix $\Phi_{m\times m}$ is an invertible matrix, the Haar coefficient vector C_m^T can be gotten by:

$$C_m^T = \hat{f}_m \ \Phi_{m\times m}^{-1} .$$

IV. Operational Matrix of the Fractional Order Integration

The integration of $H_m(t)$ defined as in eq.(3) can be approximated by Haar series with Haar coefficients P:

$$\int_0^t H_m(\tau)\, d\tau \approx P_{m\times m} H_m(t)$$

where the m-square matrix P is called the Haar wavelet operational matrix of integration, [26].

Our purpose is to derive the Haar wavelet operational matrix of the fractional order integration. For this purpose, we recall the definition (1) of fractional order integration, which is named as Riemann-Liouville fractional integration, as following:

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} f(\tau)\, d\tau$$

$$= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t)$$

where $\alpha \in \mathbb{R}$ is the order of integration, $\Gamma(\alpha)$ is the gamma function and $t^{\alpha-1}*f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$.

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Now, if $f(t)$ is expanded in Haar functions, as shown in eq.(2), the Riemann-Liouville fractional order integration becomes:

$$(I^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t) \approx C_m^T \frac{1}{\Gamma(\alpha)} \{t^{\alpha-1} * H_m(t)\}$$

Then if $t^{\alpha-1}*f(t)$ can be integrated and expanded in Haar functions, the Reimann-Liouville fractional order integration is solved via the Haar functions.

Also, we define an m-set Block Pulse Functions (BPF) as:

$$b_i(t) = \begin{cases} 1, & \frac{i}{m} \le t < \frac{i+1}{m} \\ 0, & \text{otherwise} \end{cases}$$

where $i = 0, 1, \ldots, m-1$. The functions $b_i(t)$ are adjoint and orthogonal, that is:

$$b_i(t)\, b_\ell(t) = \begin{cases} 0, & i \ne \ell \\ b_i(t), & i = \ell \end{cases}$$

$$\int_0^1 b_i(\tau) b_\ell(\tau)\, d\tau = \begin{cases} 0, & i \ne \ell \\ 1/m, & i = \ell \end{cases}$$

Because the Haar functions are piecewise constant, it may be expanded into m-term block pulse functions (BPF) as:

$$H_m(t) = \Phi_{m\times m} B_m(t) \tag{4}$$

where $B_m(t) \square\ [b_0(t)\ \ b_1(t)\ldots b_i(t)\ldots b_{m-1}(t)]^T$.

The block pulse operational matrix of the fractional order integration F^{α} is defined as follows:

$$(I^{\alpha}B_m)(t) \approx F^{\alpha}B_m(t) \tag{5}$$

where:

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha-2)} \begin{vmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \cdots & \xi_{m-3} \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Next, we derive the operational matrix of the fractional order integration set.

$$(I^{\alpha}H_m)(t) \approx P^{\alpha}_{m\times m} H_m(t) \tag{6}$$

where the m-square matrix $P^{\alpha}_{m\times m}$ is called the Haar wavelet operational matrix of the fractional order integration. Using eqs.(4) and (5), we have:

$$\begin{aligned}(I^{\alpha}H_m)(t) &\approx (I^{\alpha}\Phi_{m\times m}B_m)(t) \\ &= \Phi_{m\times m}(I^{\alpha}B_m)(t) \approx \Phi_{m\times m}F^{\alpha}B_m(t)\end{aligned} \tag{7}$$

from eqs.(6) and (7), we get:

$$P^{\alpha}_{m\times m} H_m(t) = P^{\alpha}_{m\times m} \Phi_{m\times m} B_m(t) = \Phi_{m\times m} F^{\alpha} B_m(t)$$

Then, the Haar wavelet operational matrix of fractional order of integration $P^{\alpha}_{m\times m}$ is given by:

$$P^{\alpha}_{m\times m} = \Phi_{m\times m} F^{\alpha}\, \Phi^{-1}_{m\times m}$$

V. The Approach

In this paper, we shall consider the problem of exterimization of a functional J of the form:

$$J[y(t)] = \int_{t_0}^{t_1} F[t, y(t), {}_0D_t^{\alpha} y(t)]\, dt \tag{8}$$

satisfying the condition $y(t_0) = y_0$,and $y(t_1)$ is considered to be undetermined where ${}_0D_t^{\alpha} y(x)$ is the Riemann-Liouville fractional derivative. The regular method for solving problem (8) is through the Euler equation [27]:

$$\frac{\partial F}{\partial y} + {}^cD_t^{\alpha} \frac{\partial F}{\partial D^{\alpha} y} = 0$$

and

$$\left(\frac{\partial F}{\partial {}_0D_t{}^{\alpha} y} \right)\Bigg|_{t=t_1} = 0$$

where ${}^cD_t^{\alpha}$ is the Caputo fractional derivative.

This paper mainly uses Haar wavelets to establish the direct method for fractional variational problems.

Unlike other direct methods, beginning with the assumption of the variable itself, the method we have stated here is like the method used by [28] by assuming ${}_0D_t^{\alpha} y(t)$ as Haar wavelets whose coefficients are to be determined:

$$ {}_0D_t^{\alpha} y(x) = \sum_{i=0}^{\infty} c_i h_i(t) \tag{9}$$

Taking finite terms as an approximation, we have:

$$ {}_0D_t^{\alpha} y(x) \approx \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T H_m(t) \tag{10}$$

Applying I^{α} to the both sides of eq.(10), thus $y(t)$ can be expressed as:

$$y(t) \approx C_m^T P_{m\times m}^{\alpha} H_m(t) + y(t_0) \tag{11}$$

The other terms in the functional of eq.(8) are known functions of the independent variable t and can be expanded into Haar wavelets through substitution, and finally we have:

$$J = J(c_0, c_1, \ldots, c_{m-1}) \tag{12}$$

The original extremiation of a fractional problem shown in eq.(8) becomes the extremiation of functional of a finite set of variables in eq.(12).

Taking partial derivatives of J with respect to c_i, and setting them equal to zero, we obtain:

$$\frac{\partial J}{\partial c_i} = 0, \; i = 0, 1, \ldots, m-1 \tag{13}$$

solving for c_i, and substituting into eq.(11), we have the desired result.

VI. ILLUSTRATIVE EXAMPLES

In this section, we shall introduce some examples in order to confirm the reliability of the proposed method.

Example (1):

Consider the functional:

$$J[y(t)] = \int_0^1 \left[\frac{1}{2} ({}_0D_t^{\alpha} y(t))^2 - y(t) \right] dt \tag{14}$$

and the boundary condition:

$$y(0) = y_0 \text{ and } y(1) \text{ is unspecified} \tag{15}$$

Consider that $0 < \alpha < 1$, and for solving this problem by the direct Haar wavelet method, we assume that ${}_0D_t^{\alpha} y(t)$ can be expanded in terms of Haar wavelet, as follows:

$$ {}_0D_t^{\alpha} y(t) \approx \sum_{i=0}^{m-1} c_i h_i(t) = C_m^T \mathrm{H}_m(t) \tag{16}$$

where:

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$C_m = [c_0\ \ c_1\ \ \ldots\ c_{m-1}]^T$

$H_m(t) = [h_0(t)\ \ h_1(t)\ \ldots\ h_{m-1}(t)]^T$

Here, we shall consider $m = 8$ and more accurate results may be obtained using large m.

Now, upon taking the fractional Riemann-Liouville integration to the both sides of eq.(16), thus we get:

$$y(t) = C_m^T\ P_{m\times m}^{\alpha}\ H_m(t) + y_0 \tag{17}$$

The other condition according to [27], that we have is:

$$\left.\left(\frac{\partial F}{\partial_0 D_t^{\alpha} y(t)}\right)\right|_{t=1} = 0$$

and according to our example, we get:

$$\left.{}_0D_t^{\alpha} y(t)\right|_{t=1} = 0$$

which implies that $C_m^T H_m(1) = 0$. Therefore:

$c_0 - c_1 - c_3 - c_7 = 0$

and this gives

$$c_7 = c_0 - c_1 - c_3 \tag{18}$$

substituting eqs.(16), (17) and (18) in eq.(14) yields:

$$J[y(t)] \approx \int_0^1 \left[\frac{1}{2}\left(C_m^T H_m(t) H_m^T(t) C_m\right) - C_m^T P_{m\times m}^{\alpha} H_m(t) - y_0\right] dt$$

$$= \int_0^1 \left[\frac{1}{2}\left(C_m^T H_m(t) H_m^T(t) C_m\right) - C_m^T P_{m\times m}^{\alpha} H_m(t) - [y_0\ \ y_0 \ldots y_0]\Phi_{m\times m}^{-1} H_m(t)\right] dt$$

Therefore:

$$J[y(t)] = \frac{1}{2} C_m^T \int_0^1 H_m(t) H_m^T(t)\ dt\, C_m\ -\ C_m^T\ P_{m\times m}^{\alpha} \int_0^1 H_m(t)\, dx\ - [y_0\ y_0 \ldots y_0]\ \Phi_{m\times m}^{-1} \int_0^1 H_m(t)\, dt \tag{19}$$

and it is interesting to note that the definite integral of h_0(t) from 0 to 1 is equal to 1, while the definite integral of h_1, h_2, …, h_7 are equal to zero for $m = 8$, or:

$$\int_0^1 h_0(t)\, dt\ = 1,\ \int_0^1 h_i(t)\, dt\ = 0,\ i = 1, 2, \ldots, 7 \tag{20}$$

Hence, upon using eqs.(18) and (20) and substituting into eq.(19), we get:

$$J[y(t)] \approx \frac{1}{2} C_m^T K_{m\times m} C_m - C_m^T\ P_{m\times m}^{\alpha} \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} - [y_0\ \ y_0 \ldots y_0]\ \Phi_{m\times m}^{-1} \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} \tag{21}$$

where:

$$K_{m\times m} \approx \int_0^1 H_m(t) H_m^T(t)\ dt$$

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$$= \begin{bmatrix} I_{2\times2} & & & & 0 \\ & \frac{1}{2}I_{2\times2} & & & \\ & & \frac{1}{4}I_{4\times4} & & \\ & & & \ddots & \\ 0 & & & & \frac{2}{m}I_{\frac{m}{2}\times\frac{m}{2}} \end{bmatrix}$$

If $y_0 = 0$, then eq.(21) becomes:

$$J[y] \approx \frac{1}{2} C_m^T K_{m\times m} C_m - C_m^T P_{m\times m}^{\alpha} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Following table (1) give the approximate solution of example (1) for different values of α and compares the results for $\alpha = 1$ with the exact solution which is given as:

$y(t) = t\left(1 - \frac{t}{2}\right)$.

TABLE 1

t \ α	1	0.5	0.6	0.8	Exact for α = 1
0	0.059	0.291	0.216	0.115	0.000
0.125	0.169	0.511	0.420	0.273	0.117
0.250	0.318	0.740	0.645	0.466	0.219
0.375	0.341	0.704	0.625	0.473	0.305
0.5	0.536	0.917	0.851	0.695	0.375
0.625	0.449	0.761	0.701	0.574	0.430
0.750	0.503	0.754	0.708	0.593	0.462
0.875	0.492	0.617	0.618	0.591	0.492
1	0.492	0.617	0.618	0.591	0.500

Example (2):

Consider the functional:

$$J[y(t)] = \int_0^1 \left[{}_0D_t^{\alpha} y(t) \right]^2 + \left[t\left({}_0D_t^{\alpha} y(t) \right) \right] dt \tag{22}$$

and the boundary conditions:

$y(0) = 0$ and $y(1)$ is unspecified (23)

and for solving this example also we let:

$${}_0D_t^{\alpha} y(t) = C_m^T H_m(t) \tag{24}$$

$$y(t) = C_m^T P_{m\times m} H_m(t) + y(0) \tag{25}$$

There is a variable t involved in eq.(22) explicitly and it can be expanded into Haar series over the interval [0,1]

$t \approx d_m^T H_m(t)$ (26)

Also, the other condition that we have is:

$$2\,{}_0D_t^\alpha y(t) + t\Big|_{t=1} = 0$$

which implies that:

$$C_m^T H_m(t) = -\frac{1}{2}$$

and this gives:

$$c_7 = c_0 - c_1 - c_3 + \frac{1}{2} \quad (27)$$

and substituting eqs.(24), (26) and (27) into eq.(22), we have:

$$J[y(t)] \approx \int_0^1 \left[C_m^T H_m(t) H_m^T(t) C_m + C_m^T H_m(t) H_m^T(t) d_m \right] dt$$

$$J[y(t)] \approx C_m^T K_{m\times m} C_m + C_m^T K_{m\times m} d_m \quad (28)$$

Following table (2) gives the approximate solution of example (2) for different values of α and compares the results for $\alpha = 1$ with the exact solution, which was given in [28], as:

$$y(t) = -\frac{t^2}{4}$$

TABLE 2

x \ α	1	0.5	0.6	0.8	Exact for $\alpha = 1$
0	0.001062	−0.008	−0.006441	−0.073	0
0.125	−0.008938	−0.032	−0.026	−0.085	−0.003906
0.250	−0.029	−0.079	−0.065	−0.114	−0.016
0.375	−0.049	−0.109	−0.093	−0.138	−0.035
0.5	−0.108	−0.215	−0.189	−0.214	−0.063
0.625	−0.120	−0.216	−0.193	−0.223	−0.098
0.750	−0.186	−0.437	−0.378	−0.356	−0.141
0.875	−0.222	−0.327	−0.326	−0.367	−0.191
1	−0.222	−0.327	−0.326	−0.367	−0.250

VII. CONCLUSION AND DISCUSSION

Direct Haar wavelet method has been presented for a fractional variational problem.
The procedure considered in this paper can be considered as a generalization to the results given in [28].
From the illustrative examples, it ca be seen that this operational matrix approach can obtain accurate and satisfying results. All computational results are made by MATLAB program.

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